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CONTENTS OF VOL. VI.

Page.

1. Stereoscopy in Four Dimensions—By Dr. G. H. Bryan 1
2. On the Second Derivates of the Newtonian Potential due to a Volume Distribution having a Discontinuity of the Second Kind—By Dr. Ganesh Prasad 3
3. On Napier and his Logarithms (continued from Vol. V)—By Mr. Abinashchandra Bose 13
4. Primitive Matrices and the Primitive Degrees of a Matrix, Part I—By Dr. C. E. Cullis 33
5. Renly to Prof. Bryan's Criticism—By Dr. Syamadas Mukhopadhyay 55
6. Review—Whittaker and Watson's Modern Analysis—By Mr. Hariprasanna Banerji 57

Stereoscopy in four dimensions.*

BY

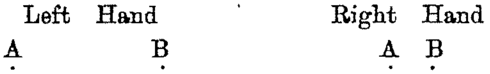
G. H. BRYAN.

The possibility of representing four-dimensional figures in a stereoscopic diagram depends on a principle different from that which is somewhat imperfectly explained by Dr. Mukhopadhyaya in a paper, entitled "A note on the stereoscopic representation of four-dimensional space" (*Bulletin of the Calcutta Mathematical Society*, *Vol. 4.*)

1. In an ordinary pair of stereoscopic pictures, near points are displaced to the left in the right-hand picture and to the right in the left-hand picture as compared with more distant ones. But it is also possible to construct diagrams in which the same point is represented a little higher or lower in one picture than the other. When seen with the two eyes, a kind of relief is produced depending on the different vertical displacements of different points and this relief represents a difference independent of the ordinary three dimensions of space. But, in order to visualise such a diagram, it is desirable that the two pictures should be simultaneously rotated so that the vertical displacement becomes horizontal after they have turned through 90°. The third and fourth dimensions will thus become interchanged. The rotation in the plane of the diagrams will therefore be accompanied by a rotation, in the plane containing the normal to the figures, and a fourth dimension of a space in which these figures are supposed to exist.

2. My method aims at visualising a stereoscopic figure in all its dimensions and, unlike Dr. Mukhopadhyaya's, it only makes use of the principle of binocular vision.

The principle involved is illustrated by the following diagrams :—



In the above diagrams, which represent the principle of an ordinary stereoscopic shell, it will be seen that the A's of the left and right hand diagrams are further apart than the B's: the result is, that when viewed through the stereoscope they look further off the observer. But now let a pair of points be represented in the two diagrams as shown below:—

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* This paper was communicated in two instalments. The first instalment consisted of the introduction and Art 1, the second consisted of Arts 2 and 3.

Left Hand	Right Hand
A.	A.
	B.
B.	

so that the height of A above B is greater in the left hand diagram.

It is clear that the picture will be seen in two different aspects by the two eyes, and that this difference cannot be associated with any difference of distance in the ordinary sense of the word. It is, however, essentially the same kind of difference as in the first diagram and the analogy leads us to associate it with a new kind of distance representing a fourth dimension in space.

3. What I said in Art 1 was that by rotating the figures round till the two representations of A B were horizontal, the difference in their distance apart then represented their distance from the observer in ordinary three-dimensional space, and it will be seen that if they be rotated still further, the one that was nearest will appear farthest. But to say that the method only gives two three-dimensional representations of a figure differing in phase by 90°, will be a complete misrepresentation.

Thus my method is superior to Dr. Mukhopadhyaya's in that it uses for the representation of a fourth dimension the same principle of binocular vision as is used for three-dimensional figures, and it is a natural corollary of that principle.

Dr. Mukhopadhyaya's method, at the best, is only successful with eyes that are able to judge distance by accommodatory power as well as by the principle of binocularity. In this respect it is only of very doubtful interest.

On the Second derivates of the Newtonian potential due to a volume distribution having a discontinuity of the second kind.

BY

GANESH PRASAD.

The object of this paper is to study, in a number of typical cases, the question of the existence of the second derivates of the Newtonian potential due to a volume distribution having a discontinuity of the *second* kind. It is believed that this question has never been discussed by any previous writer, including even Dr. Petrini in whose elaborate memoir*, "Les dérivées premières et secondes du potentiel," distributions, which are either continuous or have discontinuities of the *first* kind,† are alone considered.

1. Let O be a point inside the volume, and let the potential V be equal to u+U, where u is the potential due to a small sphere of centre O, radius a and density ρ, and U is the potential due to the remaining part of the volume Then it is obvious that at O the second derivates of U are existent; we have to consider the derivates of u.

Case I. $\rho = \cos \log \left(\frac{1}{r} \right)$

2. First let $\rho = \cos \left(\log \frac{1}{r}\right)$, where r is the distance between O and the point P where the density is ρ Then it will be shown that all the second derivates are non—existent at O, with the exception of those of the form $\dfrac{\delta^2 V}{\delta r_1 \, \delta r_2}$, r_1 and r_2 being perpendicular radii.

Divide the sphere into thin concentric spherical shells. Then by Newton's theorem relating to the attraction of a spherical shell, at O, $\dfrac{\delta u}{\delta r} = 0$, and at P

$$\frac{\delta u}{\delta r} = - \frac{4\pi \int_0^r t^2 \cos\left(\log \frac{1}{t}\right) dt}{r^2}$$

---

* See *Acta Mathematica*. Vol. 31, (1908), pp. 127—332

† See *e. g.* Art. 2 of his memoir.

Therefore, at O, $\frac{\delta^2 u}{\delta r^2}$ is existent or not according as

$$\lim_{r=0} \frac{\displaystyle\int_0^r t^2 \cos\left(\log \tfrac{1}{t}\right) dt}{r^3}$$

is existent or not.

But, putting $\log \frac{1}{t}=v$, we have

$$\int_0^r t^2 \cos\left(\log \tfrac{1}{t}\right) dt = \int_{\log \frac{1}{r}}^{\infty} e^{-3v} \cos v \, d\, v$$

$$= \left[\frac{e^{-3v}(\sin v - 3\cos v)}{1+3^2}\right]_{\log \frac{1}{r}}^{\infty} = \frac{r^3\{3\cos\left(\log \frac{1}{r}\right) - \sin\left(\log \frac{1}{r}\right)\}}{1+3^2}$$

Therefore $\displaystyle\lim_{r=0} \frac{\displaystyle\int_0^r t^2 \cos\left(\log \tfrac{1}{t}\right) dt}{r^3}$ is non-existent.

Thus it is proved that $\frac{\delta^2 u}{\delta r^2}$ and, therefore, also $\frac{\delta^2 V}{\delta r^2}$ are non-existent at O; as regards $\frac{\delta^2 u}{\delta r_1 \, \delta r_2}$, it is obviously zero, and therefore $\frac{\delta^2 V}{\delta r_1 \, \delta r_2}$ is existent.

Case II. $\rho=\cos \frac{1}{r}$.

3. Proceeding as in Case I, we find that, at O, $\frac{\delta^2 u}{\delta r^2}$ is existent or not according as

$$\lim_{r=0} \frac{\displaystyle\int_0^r t^2 \cos \tfrac{1}{t} \, dt}{r^3}$$

is existent or not.

But, putting $\frac{1}{t}=v$, we have

$$\int_0^r t^2 \cos \tfrac{1}{t} \, dt = \int_{\frac{1}{r}}^{\infty} \frac{\cos v}{v^4} \, dv.$$

Now $\frac{1}{v^4}$ is always positive and constantly diminishes as v increases. Therefore

$$\left| \int_{\frac{1}{r}}^{\infty} \frac{\cos v}{v^4}\, dv \right| < r^4.$$

Therefore

$$\operatorname*{Lim}_{r=o} \frac{\displaystyle\int_0^r t^2 \cos \tfrac{1}{t}\, dt}{r^3} = o.$$

Thus it is proved that $\frac{\delta^2 u}{\delta r^2}$ and, therefore, also $\frac{\delta^2 V}{\delta r^2}$ are existent at O. Hence it is obvious that all the second derivates of V are existent at O.

Case III. $\rho = \cos \chi(r)$.

4. I proceed now to consider the general case in which

$$\rho = \cos \chi(r),$$

$\chi(r)$ being* $\succ 1$, and to prove that $\frac{\delta^2 V}{\delta r^2}$ is non-existent at O if $\chi(r) \prec \log \frac{1}{r}$, and existent if $\chi(r) \succ \log \frac{1}{r}$. The case in which $\chi(r) = \log \frac{1}{r}$ has been already dealt with.

Proceeding as in Case I, we find that, at O, $\frac{\delta^2 u}{\delta r^2}$ is existent or not according as

$$\operatorname*{Lim}_{r=o} \frac{\displaystyle\int_0^r t^2 \cos \chi(t)\, dt}{r^3}$$

is existent or not.

But, putting $\chi(t) = v$, we have

$$\int_0^r t^2 \cos \chi(t)\, dt = \int_{t=o}^{t=r} \frac{t^2}{\chi'(t)} \cos v\, d\, v.$$

---

* The symbols $\succ$ and $\prec$ are those of the Infinitary Calculus of Du Bois Reymond, so that $f(x) \succ$ or $\prec F(x)$, according as the limiting value of $\frac{f(x)}{F(x)}$ is infinity or zero.

Now $-\frac{t^2}{\chi'(t)}$ is always positive and constantly diminishes as t diminishes. Therefore

$$\left|\int_{t=0}^{t=r}\frac{t^2}{\chi'(t)}\cos v\,d\,v\right| < \frac{r^2}{-\chi'(r)}.$$

5. If $\chi(r) \succ \log\frac{1}{r}$ and, consequently, $\chi'(r) \succ \frac{1}{r}$, it follows from the above inequality that

$$\left|\int_0^r t^2\cos\chi(t)\,dt\right| \prec r^3;$$

hence in this case

$$\operatorname*{Lim}_{r=0}\frac{\int_0^r t^2\cos\chi(t)\,dt}{r^3}=0,$$

and, consequently, $\frac{\delta^2 u}{\delta r^2}$ and, therefore, also $\frac{\delta^2 V}{\delta r^2}$ are existent at O.

6. If $\chi(r) \prec \log\frac{1}{r}$, consider the behaviour of

$$\int_{t=0}^{t=r}\frac{t^2}{\chi'(t)}\cos v\,d\,v$$

as r tends to zero by assuming the values

$$\frac{h}{m},\quad \frac{h}{m+1},\quad \frac{h}{m+2},\ \text{etc.},$$

where

$$v_{m+n}=\chi\left(\frac{h}{m+n}\right)=(m+n)\pi-\frac{\pi}{2}.$$

Now, putting

$$-\frac{t^2}{\chi'(t)}=R, \text{ and}$$

$$I_m=\int_{t=0}^{t=h_m}\frac{t^2}{\chi'(t)}\cos v\,d\,v,$$

we have

$$I_m = \int_{v_m}^{\infty} R \cos v \, dv = \int_{v_m}^{v_{m+1}} + \int_{v_{m+1}}^{v_{m+2}} + \dots$$

$$= (-1)^m \left(R_m - R_{m+1} + R_{m+2} - \dots \right),$$

where R_{m+n} is the value of R for some value of v between v_{m+n} and v_{m+n+1}.

But

$$R_m > R_{m+1} > R_{m+2} \text{ etc.}$$

Therefore I_m has the sign of $(-1)^m$ and is, numerically, greater than $R_m - R_{m+1}$.

But, since $\chi(t) \prec \log \frac{1}{t}$,

$$\chi'(t) \prec \frac{1}{t}.$$

Therefore

$$\frac{t^2}{\chi'(t)} \succ t^3.$$

Therefore

$$\frac{R_m - R_{m+1}}{h_m^3}$$

increases indefinitely as m increases

Thus, it is proved that, as r assumes the values h_m, h_{m+1}, h_{m+2}, etc., $\frac{I_m}{h_m^3}$ fluctuates in sign and increases indefinitely in magnitude. Therefore

$\frac{I_m}{h_m^3}$ does not tend to any limit as m increases; hence

$$\operatorname*{Lim}_{r=0} \frac{\int_0^r t^2 \cos \chi(t) \, dt}{r^3}$$

must be non-existent.

Therefore $\frac{\delta^2 u}{\delta r^2}$ and, consequently, $\frac{\delta^2 V}{\delta r^2}$ are non-existent at O.

Case IV $\rho=\cos\left(\frac{1}{2}\log\frac{1}{x^2}\right)$.

7. Taking O as the origin of co-ordinates, let the density at any point (x, y, z) be $\cos\left(\frac{1}{2}\log\frac{1}{x^2}\right)$. Then it will be shown that $\frac{\delta^2 V}{\delta x^2}$ is non-existent at O.

Divide the sphere into thin discs perpendicular to the x—axis. Then it is obvious that $\frac{\delta u}{\delta x}=o$ at O and that, at any point P, on the x—axis, distant x from O,

$$\frac{1}{2\pi}\;\frac{\delta u}{\delta x}=\int_{x}^{a}\left(1-\frac{t-x}{\sqrt{a^2+x^2-2tx}}\right)\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt$$

$$-\int_{-x}^{a}\left(1-\frac{t+x}{\sqrt{a^2+x^2+2tx}}\right)\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt.$$

$$=-\int_{-x}^{x}\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt-\int_{-a}^{a}\frac{t-x}{\sqrt{a^2+x^2-2tx}}\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt.$$

But

$$\frac{1}{\sqrt{a^2+x^2-2tx}}=\frac{1}{a}+\frac{x}{a^2}\,P_1\left(\frac{t}{a}\right)+\frac{x^2}{a^3}\,P_2\left(\frac{t}{a}\right)+\ldots\text{ to infinity,}$$

and, consequently,

$$\frac{t-x}{\sqrt{a^2+x^2-2tx}}=\frac{t}{a}+\frac{x}{a^2}\left(\frac{t^2-a^2}{a}\right)+\text{ terms containing higher powers}$$

of x.

Therefore

$$\frac{1}{2\pi}\;\frac{\delta u}{\delta x}=-\int_{-x}^{x}\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt-\int_{-a}^{a}\frac{t}{a}\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt.$$

$$-\frac{x}{a^2}\int_{-a}^{a}\frac{t^2-a^2}{a}\cos\left(\tfrac{1}{2}\log\frac{1}{t^2}\right)dt\text{—terms containing higher}$$

powers of x.

Now it is clear that, in the above equation, the second integral is zero, the third has a definite value, and the terms involving higher powers of x have zero as the value of the differential coefficient of their sum.

Therefore, at O, $\frac{\delta^2 u}{\delta x^2}$ is existent or not according as

$$\operatorname*{Lim}_{x=0} \frac{1}{x} \int_{-x}^{x} \cos\left(\tfrac{1}{2} \log \frac{1}{t^2}\right) dt$$

is existent or not.

But, putting $\log \frac{1}{t} = v$, we have

$$\int_{-x}^{x} \cos\left(\tfrac{1}{2} \log \frac{1}{t^2}\right) dt = 2 \int_{0}^{x} \cos\left(\log \frac{1}{t}\right) dt$$

$$= \int_{\log \frac{1}{x}}^{\infty} e^{-v} \cos v dv = 2 \left[\frac{e^{-v} (\sin v - \cos v)}{2} \right]_{\log \frac{1}{x}}^{\infty}$$

$$= x \left\{ \cos\left(\log \frac{1}{x}\right) - \sin\left(\log \frac{1}{x}\right) \right\}.$$

Therefore

$$\operatorname*{Lim}_{x=0} \frac{1}{x} \int_{-x}^{x} \cos\left(\tfrac{1}{2} \log \frac{1}{t^2}\right) dt.$$

is non-existent.

Thus it is proved that $\frac{\delta^2 u}{\delta x^2}$ and, therefore, also $\frac{\delta^2 V}{\delta x^2}$ are non-existent at O.

Case V. $\rho = \cos \frac{1}{r}$.

8. Proceeding as in Case IV, we find that, at O, $\frac{\delta^2 u}{\delta x^2}$ is existent or not, according as

$$\operatorname*{Lim}_{x=0} \frac{1}{x} \int_{0}^{x} \cos \frac{1}{t} dt.$$

is existent or not.

But, putting $\frac{1}{t}=v$, we have

$$\int_{o}^{x} \cos\frac{1}{t}\,dt = \int_{\frac{1}{x}}^{\infty} \frac{\cos v}{v2}\,dv.$$

Now $\frac{1}{v^2}$ is always positive and constantly diminishes as v increases. Therefore

$$\left| \int_{\frac{1}{x}}^{\infty} \frac{\cos v}{v^2}\,dv \right| < x^2.$$

Therefore

$$\operatorname*{Lim}_{x=o} \frac{1}{x} \int_{o}^{x} \cos\frac{1}{t}\,dt = o.$$

Thus it is proved that $\frac{\delta^2 u}{\delta x^2}$ and, therefore, also $\frac{\delta^2 V}{\delta x^2}$ are existent at O.

Case VI. $\rho = \cos\chi(x)$.

9. Proceeding as in Art. 7, we find that, at O, $\frac{\delta^2 u}{\delta x^2}$ is existent or not, according as

$$\operatorname*{Lim}_{x=o} \frac{1}{x} \int_{o}^{x} \cos\chi(t)\,dt.$$

is existent or not.

Again, a procedure, similar to that adopted in Case III, shows (1) that

$$\operatorname*{Lim}_{x=o} \frac{1}{x} \int_{o}^{x} \cos\chi(t)\,dt = o,$$

if $\chi(x) \succ \log\frac{1}{x}$, and (2) that, if $\chi(x) \prec \log\frac{1}{x}$,

$$\frac{1}{h_m}\int_{0}^{h_m}\cos\chi(t)\,dt.$$

fluctuates in sign and increases indefinitely in magnitude as m increases.

Hence $\frac{\delta^2 u}{\delta x^2}$ and, consequently, also $\frac{\delta^2 V}{\delta x^2}$ are existent when $\chi(x) \succ \log\frac{1}{x}$, and non-existent when $\chi(x) \prec \log\frac{1}{x}$.

John Napier—His Life and Work.

(Continued.)

BY

A. C. BOSE.

III. The Evolution of the Calculus of Logarithms.

THE words of Glaisher *viz.* "I cannot get over the *romance* of the invention *i.e.* it took place *when* it did and *where* it did—so much before it was due mathematically from the train of thought which led to it" will be appreciated only when we take a brief survey of the evolution of the Science and Art of computation and the stage it reached in the second half of the 16th century before the discovery of the Logarithms. What is known as the Dark Age, had closed about the end of the eleventh century and the period between this date and the fall of Constantinople and its capture by the Turks in 1453—known as the Middle Age—has been well described recently (by Whetham in "the Science and the Human mind") as "the great dark valley across which mankind had to struggle down the descent on one side from the heights of the thought of Greece and the power of Rome to the slow ascent on the other to the upward slopes of modern science. We look across the cloud-filled hollow and see the hills beyond more clearly than the intervening ground, lit only by the dim light of Scholasticism and theology." One of the foremost torch-bearers who illumined the "cloud-filled hollow" was Roger Bacon who was born about the beginning of the 13th century and whom Oxford has recently honoured by erecting his statue and commemorating his life and work. He preached that mathematics must underlie other studies and that mathematical tables and instruments were necessary. He was deeply interested in Optics, gave a theory of the rain-bow and described many mechanical inventions—mechanically driven ships and carriages and flying-machines. As Friar Bacon was great as the precursor of the age of experiment and observation—the age of Tycho Brahe, Kepler, Galileo, Napier and Newton,—Leonardo of Pisa, Fibonacci as he was called, was equally great as the man to whom we owe the "first renaissance of mathematics on the Christian soil." As a boy he learned the use of the Abacus, and as he grew older, acquired such an enthusiasm for mathematics, that in the course of business travels in Egypt, Syria, Greece and Italy, he gathered from the various peoples he met, what was then known about Mathematics. The fruit of his researches, was the Liber Abaci published in 1202, which contained about all the knowledge which the Arabs possessed in Arithmetic and Algebra. Leonardo was the first to advocate the adoption of the "Arabic Notation." Calculation with Zero was received with favour by the mathematicians on Christian soil but Leonardo had no definite symbolism to offer and the stage reached was what is known as the Stage of Syncopated Algebra which replaced the Rhetorical Algebra of the ancients. The Hindus possessed a Symbolism which had been ignored by the Arab Mathematicians. It is in works of the next great Italian Mathematician Lucas Paciuolo or De Burgo that symbols begin to appear and it is not till we come to the age of the great French Mathematicians Vieta and Descartes that we find a purely symbolic notation. As Sir John Herschel has said "our present notation has arisen by almost insensible degrees as convenience suggested different marks of abbreviation to different authors; and that perfect symbolic language which addresses itself solely to the eye, and enables us to take in at a glance the most complicated relations of quantity, is the result of a large series of small improvements." In particular, wonderful is the story of the Cipher. "Men struggled through centuries of intelligence with cumbersome number notation. At first five men in a row to count 10,000. Ten fingers raised by the first, one by the second; ten raised again by the first and two by the second: ten raised by the second and then one by the third and thus to one finger of the fifth man equalling 10,000. Then the Abacus with pebbles to represent the fingers and a row of grooves for the row of men. Centuries of intellect using the decimal system but struggling with it because unable to comprehend the possibility of Cipher to indicate position in the row of men or in the grooves of the Abacus; centuries of struggling with other symbols, adopting Hindu numerals, but failing to find a character which would indicate nothing—to indicate that a position was empty. A Hindu genius created Zero. Hindu numerals have been traced back to the early part of the third century B.C. "But a whole millennium was yet to pass before the creation of the most useful symbol in the world—the naught—the Zero the first known use of which in a document is in 738 A.D. This little ellipse, picture for airy nothing is an indispensable corner-stone of modern civilization. It is an Ariel lending magic powers to computation. This giving to airy nothing not merely a local habitation and a name but a picture, a symbol, is characteristic of the Hindu race whence it sprang. It is like coining Nirvana into Dynamos." This invention of a symbol for nothing, as has been said by another writer is "the crowning and transcendent achievement in the perfection of the Decimal system and lies at the base of all subsequent arithmetical progress."

The powers of the Cipher so eloquently described, developed still further in the Art of computation when its linear dimensions vanished and the decimal point was evolved in the course of succeeding centuries and Napier, with Simon Stevin of Bruges, was one of the mathematicians first to recognise its potentiality as an operative form.

Italy, then, was the country in which the Renaissance began—Italy which had buried Rome and its glory and which had not yet dropped its link with Greek thought. With the brilliant beginning made with the Liber Abaci, the Practica Geometria and the Flos of the great Leonardo, and the spread of mathematical culture throughout Italy, one should have thought that Mathesis would enjoy a steady and vigorous development throughout Europe. Not so was the case, for during the 14th and 15th centuries, long wars absorbed the energies of the people. The death of Frederick II in 1254 led to confusion in Germany and to constant quarrels between the German Emperors and the Popes. Italy was drawn into the vortex and knew no peace between the Guelphs and the Ghibellines. France and England closed with each other in the Hundred Years' War (1338—1453) and England herself was torn by the Wars of the Roses. To these wars was added the benumbing influence of Scholasticism which revelled in hair-splitting arguments, leading to indistinctness and confusion of ideas, for an adequate and learned account of which the reader is requested to refer to the pages of Whewell's monumental work on the History of the Inductive Sciences. But the Torch once lit was not to be put out and the invention of printing about 1450, facilitated the communication of knowledge from centre to centre till it reached the remotest corners of Europe. Looking into the past history of mathematics, do we not find Cuthbert Tonstall (1474-1559), proceed to Padua after a course of study in Cambridge and Oxford? It is said that "he won there a high and well-deserved reputation, distinguishing himself more particularly in Jurisprudence. But during his stay in Italy he was also deeply interested in mathematics and made a careful study of the works of Paciuolo." Great was his enthusiasm for the Science of Algebra and in his work *De Arte Supputandi Libri quattuor* (1522) he set forth its principles as learnt in "beautiful Italy." It was, however, in the German University of Prague, founded at the end of the 14th century, that greater attention was paid to the cultivation of mathematics than in the older Universities of Paris, Bologna, Padua and

Pisa where lectures on Astrology still took the place of lectures on Ptolemy's Almagest. And it is to the great German mathematicians of the 15th and 16th centuries George Purbach, John Mueller (Regiomontanus), Michæl Stifel and George Joachim (Rhæticus) that we owe those great developments of the Science and Art of computation which made it such a powerful instrument in the study of the heavens and a material help to the labours of the great astronomers of the sixteenth century. Remarkable, in particular, is the work, Arithmetica Integra (1544) of Michæl Stifelus or Stifel, the great German Algebraist, who was drawn to mathematics by "a study of the significance of mystic numbers in Revelation and in Daniel." Melanchthon wrote a preface to the work which is divided into three parts dealing respectively with rational numbers, irrational numbers and Algebra. The advances which he made beyond the stages reached by the Italian mathematicians of the 16th century have been summed up by Dr. Hutton as follows :—

(1) He introduced the characters, +, −, √ for plus, minus, root or radix.

(2) He introduced the initials 2, 3, 4, ——— for the powers.

(3) He treated all the higher order of quadratics by the same general rule and he used the general literal notation a, b, c, —— for so many different unknown or general quantities.

(4) *(And this is most important for the purposes of this essay).*—He introduced the numeral exponents of the power, −3, −2, −1, 0, 1, 2, 3, etc. both positive and negative, so far as integral numbers were concerned but *not fractional ones*, calling them exponents and he taught the general uses of the Exponents, in the several operations of powers, as we now use them or the logarithms. Here, then are the germs of the theory of Exponents, which was not fully developed till about the time of Newton, *i.e.*, long after Napier's invention. To Stifel, however, is due the great merit of showing "the relations between operations with the terms of a Geometric and an Arithmetic series, in which the terms are made to *correspond*, viz. the relations between multiplication, division and exponentiation on the one hand and addition, subtraction, multiplication or division on the other" relations which really contained in part the theory of logarithms.

With Stifel then we arrive at an advanced stage of the Arithmetical and Algebraical sciences and if we added the work of Simon Stevin of Bruges, on decimal fractions, and the productions of Robert Recorde in England (who introduced the sign of equality) whose *Grounde of artes* and *Whettstone Witte* were referred to in their writings even by Shakespeare and Sir Walter Scott, so widely known and used, were these works on Arithmetic and Algebra, we place before the reader a conspectus of the state of the Science and Art of computation on the arithmetical side before the invention of logarithms. We find about the middle of the 16th century, a fairly complete symbolic Algebra with the first three fundamental laws of operation fully developed and the fourth—the law of indices—only partially developed. The functional nature of a^x had, however, not been recognised and the theory of infinite series remained to be worked out. On the other hand, we find an Arithmetic the same as at the present date, with all the instruments of computation nearly ready. On the side of Trigonometry—a Science which progressed side by side, with Arithmetic and Algebra, we note that in the fruits of the labours of George Purbach and his pupil—Regiomontanus, and of Rhæticus, we find those developments of the Art of computation (the Trigonometrical tables) which eventually necessitated the invention of the Logarithms. The first table of sines of Germanic origin is of Purbach. He put the radius equal to 6,00,000 and computed at intervals 10, Regiomontaus computed two new tables of sines, one for $r=6,000,000$ and the other for $v=10,000,000$. Besides these he constructed a table of tangents for every degree, taking $r=10^5$. In this field of mathematical tables we must note the great labours of Joachim Rhæticus. He calculated the first table of secants and also tables of sines and tangents for every $10''$ for $r=10^{15}$. His work was monumental and when published by Valentine Otho (under the title Opus Palatinum) about the end of the 16th century filled a volume of 1468 pages ! ! The labour involved in the calculation of these tables, in the absence of the Calculus of Logarithms, can be better imagined than described and we are told that Rhæticus had to employ for this purpose "a number of computers for twelve years and spent thereby thousands of Gulden". Bartholomew Pitiscus republished these great tables in 1613 (the year before the publication of Napier's invention) freeing them from errors as far as possible. The tables of Pitiscus contain values of sines, tangents and secants on the left and of the complements on the right. He added proportional parts for every minute and even for every ten seconds and in the whole calculation r is taken as 10^{15}. As remarked by a competent authority "the enormous work undertaken by Rhæticus needs no eulogy; and the earnestness and love of accuracy displayed by Pitiscus, not only rendered apparent by his acts but also evident in the prefaces to his several works, will always render his an honoured name in Science". We have thus arrived at the stage at which it may be said that the tables of numerical values of the trigonometric functions had attained a high degree of accuracy but at what an enormous cost of time and labour! Naturally at the end of the 16th century, mathematicians turned their earnest attention to the devising of a calculus by which the time and labour devoted to these enormous calculations—calculations absolutely necessary for astronomical and other purposes, could be economised. Prof. Playfair has written in his well-known Dissertation on the progress of the Mathematical Sciences. "Napier whose mind seems to have been peculiarly turned to Arithmetical recearches and who was also devoted to the study of astronomy, had early sought for the means of relieving himself and others from this difficulty. He had viewed the subject in a variety of lights and a number of ingenious devices had occurred to him by which the tediousness of Arithmetical operations, might, more or less, completely be avoided. In the course of these attempts, he did not fail to observe, that when the numbers to be multiplied or divided were terms of a geometrical progression, the product or the quotient must also be a term of that progression and must occupy a place in it pointed out by the places of the given numbers, so that it might be found from mere inspection, if the progression were far enough continued. If, for instance, the third term were multiplied by the seventh, the product must be the tenth and if the twelfth were to be divided by the fourth, the quotient must be the eighth so that multiplication and division of such terms was reduced to to the addition and subtraction of the numbers which indicated their places in the progression".

"This observation or one very similar to it was made by Archimedes and was employed by that great geometer to convey an idea of a number too vast to be correctly expreseed by the Arithmetical notation of the Greeks (vide pp. 221—232 of the Works of Archimedes—edited by Sir Thomas Heath). Thus far, however, there was no difficulty and the discovery might certainly have been made by men inferior either to Napier or Archimedes. What remained to be done, what Archimedes did not attempt (and we might add, what even Stifel did not discover—although he came close to it) and what Napier completely performed, involved two great difficulties. It is plain that the resource of the Geometrical Progression was sufficient when the given numbers were terms of that Progression, but if they were not, it did not seem that any advantage could be devised from it. Napier, however, perceived, and it was by no means obvious, that *all numbers whatsoever might be inserted in the Progression and have their places assigned in it.* After conceiving the possibility of this, the next difficulty was to discover the principle and to execute the *Arithmetical process,* by which these places were to be ascertained. It is in these two points that the peculiar merit of his invention consists; and at a period when the nature of series and when every other resource of which he could avail himself, were so little known, his success argues a depth and originality of thought, which have rarely been surpassed. The way in which Napier satisfied himself that all numbers might be intercalated between the terms of the given G. P. and by which he found the places they must occupy, was founded on a most ingenious (Kinematical) supposition—viz., that of two points describing two different lines—the one with a constant velocity and the other with a velocity always increasing in the ratio of the space the point had already gone over; the first of these would generate magnitudes in Arithmetical and the second magnitudes in Geometrical progression. It is plain, that all numbers whatsoever would find their places among the magnitudes so generated (because of the linear continuum) and indeed, this view of the subject is as simple and profound as any which has yet presented itself to mathematicians."

"The numbers which indicate (in the A.P.) the places of the terms of the G.P. are called by Napier, the logarithms of those terms."

"Various systems of logarithms, it is evident, may be constructed, according to the G. P. assumed."

"If Logarithms had not been invented by Napier, they would have been discovered in the progress of Algebraic analysis, when the powers and exponents, both integral and fractional, came to be fully understood. The idea of considering all numbers as powers of one given number would then have readily occurred and the doctrine of series would have greatly facilitated the calculations which it was necessary to undertake. Napier had none of the advantages (as will be seen by the state of knowledge in the Science and Art of computation in Napier's time mentioned above) and they were all supplied—*by the powers of his mind.* Napier's good fortune, also, not less than his great sagacity, may be remarked. Had the invention been delayed to the end of the 17th century, it would have come about *without effort* and would not have conferred on the author the high celebrity which Napier so justly derives from it."

Napier's great work was published by Andrew Hart, book-seller, in 1614 in Latin under the title "Mirifici Logarithmorum Canonis Descriptio." This work which will be briefly called hereafter the Descriptio, contains the natural sines and the logarithms of the sines for every minute of the quadrant with a description and explanation of the uses of the Tables. I have not the work before me and I follow Professor Hobson in describing the contents of this work:—

There are 57 pages of explanatory matter and 90 pages of Tables. The explanatory matter contains an account of, Napier's conception of a logarithm, their principal properties and their application in the solution of plane and spherical triangles. Napier's well-known rules of circular parts containing the complete system of formulæ for the solution of right-angled spherical triangles are here given. To understand this work, it should be remembered that the sine of an angle was not, in Napier's time and long afterwards, regarded as a ratio but as the length of that semichord of a circle of given radius which subtends the angle at the centre. Napier took the radius to consist of 10^7 units and thus the sine of 90° called the whole sine, is 10^7, the sines of smaller angles decreasing from this value to zero. The Table is, therefore, one of logarithms of numbers between 10^7 and zero, not for equidistant numbers but for numbers corresponding to equidistant angles.

The Descriptio was welcomed by the greatest mathematicians as giving once for all the long desired relief from the labour of calculation. The immortal Johannes Kepler "who was engaged in the prodigious task of discovering and verifying by numerical calculation, the laws of motion of the planets" in the early years of the 17th century, hailed with delight this great invention. He was one of the first in Germany to appreciate the full importance of the facilities, logarithms afford to the numerical calculator. Writing in 1618 to his friend Schick-hard says Kepler "there is a Scotish Baron (whose name has escaped my memory) who has made a famous contrivance by which all need of multiplication and division is supplied by mere addition and subtraction but he does it without sines. But even he wants a table of tangents and the variety, frequency and difficulty of the additions and subtractions in some cases is greater than the labour of multiplying and dividing."

Kepler dedicated his "Ephemeris" for 1620 to the author of this celebrated invention, Baron Napier of Merchistoun (not knowing that he had died in 1617) and in 1624 himself published a book of Logarithms, entitled Chilias Logarithmorum containing the Napierian logarithms of the quotients of 10^5 divided by the first ten numbers, then proceeding by the quotients of every ten to 100 and by hundreds to 100,000. In the supplement published in 1625 there is a curious notice by Kepler of the manner in which this "subtle contrivance" was at first received on the Continent. It runs "In the year 1621 when I had gone into Upper Austria and had conferred every where with those skilled in mathematics, on the subject of Napier's Logarithms, I found that those whose *prudence had increased* and whose *readiness had diminished*, through age, were hesitating whether to adopt this new sort of numbers, instead of a tabe of sines; because they said it was disgraceful to a professor of mathematics to exult like a child at some compendious method of working, and meanwhile to admit a form of calculation, resting on no legitimate proof, and which at some time might entangle us in error, when we least feared it. They complained that Napier's demonstration rested on a fiction of *geometrical motion*, too loose and slippery for a sound method of reasonable demonstration to be founded on it.* (It will be seen below that it was neither loose nor slippery). This led me forthwith to conceive the germ of a legitimate demonstration, which during that same winter I attempted, without reference to lines or motion, or flow, or any other which I may call sensible quality."

"Now to answer the question; what is the use of logarithms? Exactly what ten years ago was announced by their author, Napier, and which may be told in these words. Wheresoever in common arithmetic, and in the Rule of Three, come two numbers to be multiplied together, there the sum of the logarithms is to be taken; where one number is to be divided by another, the difference; and the number corresponding to this sum or difference, as the case may be, will be the required product or quotient. This, I say, is the use of logarithms. But in the same work in which I gave the demonstration of the principles, I could not satisfy the unfledged arithmetical chickens, greedy of facilities, and gaping with their beaks wide open, at the mention of this use, as if to bolt down every particular gobbet, till they are crammed with my precepticles."

But none received with more enthusiasm this grand invention of the Peer of Merchistoun than the noted contemporary mathematicians of England, Edward Wright and Henry Briggs. The name of Edward Wright has been handed down to posterity, as the greatest English authority of the 16th century on the Theory of Navigation who improved on the work of Gerard Mercator—the French mathematician and Geographer—one of the early pioneers of the modern Theory of Conformal representation of surfaces, and whose name-sake Nicholas Mercator, (although a German, he spent most of his time in England and was a member of the Royal Society of London), exposed the true theory of the Logarithms in his work Logarithmotechnia published in 1868 and discovered the series for log (1+x). Wright was a fellow of Caius and had a remarkable talent for the construction of instruments. He subsequently became lecturer on mathematics to the East India Company on the stipend of £50 a year and then private tutor to the Prince of Wales. He constructed a sphere "which enabled the spectator to forecast the motions of the solar system with such accuracy that it was possible to predict the eclipses for over seventeen thousand years in advance."

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* Similar was the objection originally made to Newton's Fluxions. Napier's idea of logarithms was identical with that method of conceiving quantities. This singular coincidence of ideas will appear if we peruse what Newton himself wrote half a century later (1665) in his *Tractus de Quadratura Curvarum.*

When in 1614 Napier's invention was announced Wright saw at once its immense value for all practical problems in Navigation and Astronomy. He took up the work of translating it into English and sent the translation to Napier who approved of it. But Wright died in 1615 and the translation was brought out in 1616 by his son, Samuel Wright, another Cambridge mathematician, under the title "A description of the admirable table of Logarithms," in which we find the description of a triangular diagram invented by Wright for finding the proportional parts.

To Henry Briggs, however, is due the credit of the rapid adoption of Napier's grand discovery by mathematicians. No name stands higher in the Annals of the Cambridge School of Mathematics than that of this ardent mathematician and calculator—the first holder of the Gresham Chair of Geometry in London and the Savilian Chair of Geometry at Oxford. These were the dark days when the study of mathematics at Cambridge was at a low ebb and mathematicians like Briggs, Pell, Wallis and Ward had to find their life's work away from it. "It seemed that there was no career for a mathematician at Cambridge" and it was not till the influence of Newton was felt that Cambridge began to assume a recognised position in the mathematical world. The Universal recognition of the value of Newton's work helped Cambridge to step out of the Slough of Despond, and to proceed on the ever-onward march to conquests of regions unknown.

IV. Napier and Henry Briggs.

Never was friendship and mutual esteem between two savants more intimate and discriminating than between the great Scotchman and the Englishman. In a letter to Archbishop Ussher dated March, 10th 1615, Briggs wrote :—"Napper, Lord of Markinston, hath set my head and hands at work with his new and admirable logarithms. I hope to see him this summer if it please God, for I never saw book which pleased me better or made me wonder. I purpose to discourse with him concerning eclipses, for what is there which we may not hope for at his hands. I am wholly employed about the noble invention of logarithms lately discovered," The ardour with which Briggs studied the *Canon Mirificus* has been described by Dr. Thomas Smith as follows :—

"Hunc in deliciis habuit, in sinu, in manibus, in pictore gestavit, oculisque avi dissimis et mente attentissima, iterum ierumque perlegit."

In the summer of 1615, Briggs visited Napier and stayed with him a full month. In happy and learned discourse these two congenial souls, discussed improvements in the Canon. The first visit of the two men has been so well described by Lilly the Astrologer and has been so often referred to (notice the address on Geometry delivered with all his matchless eloquence, by Sylvester when appointed Gresham Professor at Oxford—the chair first occupied by Briggs) that it will bear quotation here :—

"At first, when the Lord Napier first made publick his logarithms, Mr. Briggs was so surprised with admiration of them that he could have no quietness in himself, until he had seen that noble person, the Lord of Marchiston, whose only invention they were: he acquaints—John Marr (an excellent Mathematician and Geometrician, a friend of Briggs and Napier) who went into Scotland before Mr. Briggs, purposely to be there when these two learned persons should meet. Mr. Briggs appoints a certain day when to meet at Edinburgh: but failing thereof the Lord Napier was doubtful he would not come. It happened one day as John Marr and Lord Napier were speaking of Mr. Briggs; "Ah, John," said Marchiston "Mr. Briggs will not now come;" at the very instant one knocks at the gate; John Marr hasted down and it proved Mr. Briggs to his great contentment. He brings Mr. Briggs up into my Lord's chamber, when almost *one quarter of an hour was spent, each beholding the other almost with admiration, before one word was spoke.* At last Mr. Briggs began:—"My Lord, I have undertaken this long journey purposely to see your *person* and to know by what engine of wit or ingenuity, you came first to think of this most excellent help unto Astronomy, viz., the Logarithms; but my Lord, being by you found out, I wonder no body else found it out before, when now known, it is so easy."

Briggs was nobly entertained by Napier and every summer after that, during Napier's life-time, Briggs, although he was nearing 60, made his pilgrimage to Marchiston and Napier refers to him as "Amico Mihi longe Charissimo."

And right loyally did Henry Briggs discharge the duties which his admiration of the great work of Napier imposed on him. "A man of great probity, a contemner of riches, contented with his own station and preferring a studious retirement to all the splendid circumstances of his life," Briggs, before he died, had brought the calculus of logarithms to such a stage of perfection that the labours of a host of workers in the same field in the centuries that followed have added but little to the work of Briggs. Pass we lightly the criticisms of the late Dr. Charles Hutton that Napier was not fair to Briggs in regard to the improvement which consisted of the change to the decimal base. As a matter of fact Dr. Hutton's, otherwise admirable History of Logarithms, is tainted, as it seems to me, by a somewhat low estimate of Napier's merits as an inventor and a mathematician of high rank. If we turn to Briggs's own account of the matter, as given in his Arithmetica Logarithmca of 1624, we find that Briggs had suggested that "the logarithm of the tenth part of the whole sine should be 10^{10}" but Napier said that it would be better that "o should become the logarithm of unity and 10^{10} of the whole sine." Briggs wrote that he could but admit that this change was by far the most convenient of all As a result of this conference Briggs rejected the tables he had aleady prepared and commenced, as he says, "under Napier's encourageing counsel, to-ponder seriously about the calculation of these tables".

V. Calculation of Logarithms—The Constructio:

The Canon Mirificus published in 1614, did not contain an account of the *method* by which the logarithms had been calculated. Napier states that "the use and profit of the thing first conceived, the rest may please the more, being set forth hereafter, or else displease the less, being buried in silence. For I expect the judgment and censure of learned men hereupon, before the rest rashly published, be exposed to the detraction of the envious."

The explanation of the method was, however, given in the posthumous work—entitled "Mirifici Logarithmorum Canonis Constructio" which is generally briefly called "the Constructio". It was published by Napier's second son by his second marriage, Robert Napier, in 1619. Excerpts from the Preface (put as "Greeting to the Reader Studious of Mathematics") to this work written by the dutiful son—Napier's scientific executor—are given below as throwing light on the history of the construction of the wonderful Canon and especially on the part played by Henry Briggs, the co-adjutor of Napier :—

"But, since his departure from this life, it has been made plain to me by unmistakable proofs, that the most skilled in the mathematical sciences consider this new invention of very great importance, and that nothing more agreeble to them could happen, than if the Construction of this Wonderful Canon, or at least so much as might suffice to explain it, go forth into the light for the public benefit."

"You have then (kind Reader) in this little book most amply unfolded the theory of the construction of logarithms,) here called by him artificial numbers, for he had this treatise written out beside him several years before the word Logarithm was invented) in which their nature, characteristics, and various relations to their natural numbers are clearly demonstrated."

"It seemed desirable also to add to the theory an Appendix as to the construction of another and better kind of logarithms (mentioned by the Author in the preface to his Rabdologiae) in which the logarithm of unity is o."

"We have also taken care to have printed some studies on the above-mentioned Propositions, and on the new kind of Logarithms, by that most excellent Mathematician Henry Briggs, public Professor at London, who for the singular friendship which subsisted between him and my father of illustrious memory, took upon himself, in the most willing spirit, the very heavy labour of computing this new Canon, the method of its creation and the explanation of its use being left to the Inventor. Now, however, as he has been called away from this life, the burden of the whole business would appear to rest on the shoulders of the most learned Briggs, on whom, too, would appear by some chance to have fallen the task of adorning this Sparta."

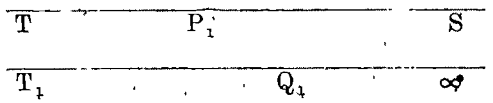
The Constructio contains, naturally, no table, being a sequel to the Descriptio. In 1873, when Dr. Glaisher wrote his celebrated Report on Mathematical Tables, the Constructio was a rare work, and the only copy which he could see was in the Cambridge University Library and had once belonged to Oughtred (1574—1660)—a contemporary Cambridge mathematician whose part in the promulgation of the Canon, I have nowhere seen precisely described—whose Clavis Mathematica (1631) introduced for the first time the sign, ×, of multiplication and the symbol ::, in Proportion, who was somewhat of an Astrologer and Alchemist" (signs of the times again !), "who complained bitterly of the penury of his wife who always took away his candle after supper, 'whereby many a good motion was lost and many a problem unsolved' and "whose warmest esteem was won by one of his pupils who secretly gave him a box of candles."

One day, some years ago, while out one morning for a walk in the streets of Calcutta I saw a collection of old books for sale on the road-side and being curious I looked into them. Great was my surprise when I discovered among much that was rubbish, W. R. MacDonald's English Translation of the Constructio, published in 1889 by Messrs. William Blackwood and Sons of Edinburgh and London—a copy of that had been presented to some one—perhaps a lover of the Science and Art of computation. I. purchased it forthwith for a rupee; although it is worth its weight in gold. Little thought I then that the Tercentenary of Napier's great invention would come in 1914 and that an endeavour would have to be made by me to give an account of it; Professor E. W. Hobson's lecture in 1914 on "John Napier and the Invention of Logarithm, 1614," has been published as a brochure of 48 pages by the Cambridge University Press and thereby one more precious production has been added to several elementary expositions of land-marks in the History of Mathematics, from the pen of the erudite student of the Theory of Functions on English soil. And Prof. H. S. Carslaw's admirable paper on the "Discovery of Logarithms by Napier," unfortunately delayed for production at the Napier Tercentenary at Edinburgh, has since appeared in the Mathematical Gazette for May and July, 1915. Both these writers have afforded an opportunity to workers with a recent Table of logarithms to learn its past history and to appreciate with delight the romance of the Invention. Even Macdonald's Translation of the Constructio (1889) is rare, and Professors Hobson and Carslaw deserve our thanks for their works. Prof. Hobson has enlivened his pages by giving glimpses of Napier's past life—an account based on the admirable articles on Napier and Logarithms in the Encyclopaedia Britannica by Dr. Glaisher and on the Biography by Marks Napier (1838).

Napier's Constructio consists of 59 main articles with an Appendix and Remarks thereon and a chapter headed "Some very remarkable propositions for the solution of spherical triangles with wonderful ease." The work opens with the definition of a Logarithmetic Table as "a small table by the use of which we can obtain a knowledge of all geometrical dimensions and motions in space by a very easy calculation (Art. 1) and ends with instructions how to form a logarithmetic Table (Art. 59). The modern Theory of limits was present in the mind of Napier when (Art. 7), he lays down that "an unknown or incommensurable quantity is included between numerical limits not differing by *many units* and that in place of the unknown quantities themselves, their limits are to be added, subtracted, multiplied and divided according as there may be need." In Arts. 8 to 11, he deals with addition, subtraction, multiplication and division of limits. In Arts. 13 to 15, he lays down rules for easily forming an A. P. and a

G. P. In Art 16, he forms his First Table—a table of 101 terms in G. P. beginning with $a_0=10^7$, the common ratio r_1 being $1-\frac{1}{10^7}$ and the last term being $a_{100}=a_0\ r_1^{100}$. In Art. 17, Napier forms his Second Table, another G. P. with the first and last terms of the first G. P. as the first two terms of the new G. P., of 51 terms, b_0, $b_1 \ldots\ldots b_{50}$, the common ratio, being r_1^{100}. (There is an error in Napier's calculation of the last term of this series, viz. that the decimal portion should have been ·224804 instead of ·222927). Napier does not calculate the the terms of this series but replaces it by one easier to compute, viz., one in which the common ratio is $1-\frac{1}{10^5}$. In Art. 18, Napier calculates another G. P. consisting of 21 terms, c_0, c_1, $c_{20} \ldots\ldots c_{20}$. The first term is as before $c_0=10^7$ and the second term very nearly the same as the last term in the second Table and the common ratio is taken as the ratio of the last to the first term of the second Table which is approximately $1-\frac{1}{2000}$. In Arts. 19 to 21, Napier forms his Third Table, with 69 columns. The terms at the top of each column form another G. P. in which the first two terms are c_0 and a number different only slightly from c_{20} (the last term of the series formed in Art. 18), the common ratio being $1-\frac{1}{10^2}$. Having thus got the *first* terms in each column, viz. $c_0\ (=10^7)$, $c_0\left(1-\frac{1}{10^2}\right)$, $c_0\left(1-\frac{1}{10^2}\right)^2 \ldots$, he obtains 21 terms in each column from its first term with the common raito $1-\frac{1}{2000}$. Napier thus obtained his Third Table—a set of numbers lying between 10^7, and very nearly $\frac{1}{2}\cdot 10^7$—"a dense enough set" for dealing with sines of angles from 90° to 30°. Napier then proceeds to deal with his kinematical definition of logarithms and to lay down two propositions about limits between which a logarithm would lie Napier takes two right lines TS and $T_1\infty$. TS is taken to be equal to 10^7 units, the radius of the circle for which the sines are measured and a point P is supposed to move from left to right so that its velocity is at every point proportional to the distance from S. At the same time he supposes a point Q moving along the line $T_1\infty$ with uniform volocity (the same as that of P initially).

Fig. 1.



Let P_1 and Q_1 be the corresponding positions of the two points; Napier defined the number representing $T_1 Q_1$ as the logarithm of the the number representing SP_1. Thus, the logarithm of the whole sine or 10^7 is zero.

Put into the language of the differential calculus, this *kinematical definition* is equivalent to the integration of the differential equation

$$\frac{dy}{dt}+cy=0,$$

Where $SP_1=y$, $T_1Q_1=X$, and the initial velocity of both particles is cr (r being the radius).

$$\text{As } X=c\,r\,t,$$

$$\text{We get } x=r\,.\log_e\left(\frac{r}{y}\right).$$

$$\text{Thus Log}_{nap}\, y=r\log_e\left(\frac{r}{y}\right)$$

Napier easily deduces from his definition that as the lengths T_1 Q_1 increase in *A. P.*, the corresponding lengths SP_1 decrease in *G. P.*.

After defining Logarithms as above, Napier deduces the following properties of products and quotients of two sines or numbers :—

$$\text{If } z=\frac{mn}{r} \text{ then } L(z)=L(m)+L(n) \quad (1)$$

$$\text{If } \frac{z}{r}=\frac{m}{n} \text{ then } L(z)=L(m)-L(n) \quad (2)$$

$$\text{If } \frac{m}{n}=\frac{m^1}{n^1} \text{ then } L(m)-L(n)=L(m^1)-L(n^1)' \quad (3)$$

[*N.B.* With Napier $L(m\cdot n)$ is not $L(m)+L(n)$, nor $L\left(\frac{m}{n}\right)$ is $L(m)-L(n)$].

Further, he obtains two remarkable theorems about the superior and inferior limits for a logarithm and for the difference of two logarithms which he made the basis of the calculation of his Canon. They are, x and y being two sines or numbers, and $x>y$:—

$$(10^7-x)\ \frac{10^7}{x}>L(x)>(10^7-x)\ldots\ldots\ldots\ldots..(4)$$

$$10^7\ \frac{(y-x)}{x}>L(x)-L(y)>\ 10^7\ \frac{(y-x)}{y}\quad(5)$$

All these theorems are now easy to establish from the index definition of the logarithm and the series for log $(1+x)$, but in Napier's day, as already stated, these were not known and Napier had to deduce them all from his kinematical definition and the propositions about limits he had established, in the strict Euclidean fashion, in his Constructio.

The rest of the construction of his Canon was plain sailing. Having obtained the limits as in (4), Napier applies it to his First Table. He takes the second number in that Table viz. 10^7—1. According to (4) its logarithm lies between......1·0000001 and 1·0000000 and Napier takes the A.M. 1 00000005 for the required logarithm. As the subsequent numbers are in G.P. with the first two, the logarithm of the last number in the First table, is the 101 st term of an A.P. whose first term is zero and common difference is 1·00000005.

Theorem (5) is employed to obtain limits for the logarithms of numbers *nearly* equal to a number in the First Table. Thus from the logarithm of the last term in the First Table, we pass, by means of—(5) to the logarithm of the Second term in the Second Table and we find by taking the A.M. of the limits given by (5), the logarithm to be 100·0005000. As the remaining 49 terms in this Table are in G. P. with the first two, the logarithm of the last number in the Second Table, being the 51st term of an A. P. whose first term is zero and common difference 100·00050C0, is of course, 5000·0250000.

Napier then passes from the last term of the Second Table to the second term of the first column of the Third Table and by similar application of theorem (5), obtains its logarithm as 5001·250417. This, then, is the common difference of the logarithms of the twenty-one numbers in the first column of the Third Table which are in G. P. We obtain thus the logarithm of the last number in the first column of the Third Table and the passage from that to the logarithm of the first number of the second column is easy and the latter is found to be 100503·358. As stated above, the top numbers are all in G. P. and we can write down their logarithms, knowing the logarithms of the first two of them. The second term in each column is a number differing slightly from the first, and again by the application of (5) we obtain the logarithm of each of the second numbers in each of the 69 columns. Thus, when we know the logarithms of the first two numbers in each column we can find the logarithms of the rest.

The results of all these calculations for the Third Table were embodied by Napier in a Table called by him the Radical Table. Both Professors Hobson and Carslaw have given this Table in their papers the same as in Macdonald's Translation of the Constructio.

The Radical Table would, thus; give the logarithms of the sines of all angles between 30° and 90°, with the help of Theorem (5). The sines or numbers contained in the Table are dense enough and intercalations would be found easy by the help of this all-important Theorem.

To obtain the logarithms of sines or numbers not included in the limits of the Radical Table, Napier gave two methods :—

(1) He calculated a Short Table containing the difference of the logarithms of two sines of which the ratio is compounded of the ratios 2 : 1 and 10 : 1

(2) He uses the formula

$$\frac{r}{2}\ \text{Sin}\ 2\theta = \sin\theta\ \text{Sin}\left(\frac{\pi}{2}-\theta\right)$$

which gives $L(\text{Sin}\ 2\theta) + L\left(\frac{r}{2}\right) = L(\text{Sin}\ \theta) + L\left[\text{Sin}\left(\frac{\pi}{2}-\theta\right)\right]$

and the table of logarithm of sines can be extended to 15° and thence to 7°—30′ and so on.

Prof. Hobson has given the Short Table and explained its use on pp. 36 and 37 of his Lecture (Vide Arts. 53 and 54 of the Constructio).

Napier thus constructed his Principal Canon of logarithms of sines of angles in the whole quadrant for $r=10^7$. And we cannot render our description of the great invention more illuminating than in the words of Lagrange "as for the rest, since the calculation of logarithms is now a thing of the past, except in isolated instances, it may be thought that the details into which we have entered are devoid of value. We may, however, justly be curious to know the trying and tortuous paths which the great inventors have trodden, the different steps which they have taken to attain their goal, and the extent to which we are indebted to these veritable benefactors of the human race. Such knowledge, moreover, is not matter of idle curiosity. It can afford us guidance in similar enquiries and sheds an increased light on the subjects with which we are employed."

As a mathematician, Napier's claims to high recognition rest on his invention of the Logarithms. Such claims have been contested on the Continent on behalf of Jobast Bürgi (1552-1632), a Swiss Watchmaker and Instrument-maker, who invented a system of Logarithms. As pointed out by Dr. Glaisher, in his article on Logarithms in the Encyclopædia Britannica Bürgi's work known as "the Arithmetische and Geometrische Progress Tabulen," the priceless Dantzic copy of which was exhibited at the Napier Tercentenary at Edinburgh, is really a Table of anti-logarithms. As remarked by Dr. Hobson, Bürgi's system is inferior to that of Napier, and the knowledge of the use of the Logarithms which was spread in the Scientific world. was entirely due to the work of Napier."

Among the other works of Napier, we need hardly deal with his "Rabdologia" which is now out of use, although "Napier's bones" were at one time in much request both in Great Britain and on the Continent. But we cannot overlook his work on Arithmetic and Algebra known as "De Arte Logistica," a work which Napier had to lay by in order that he might devote himself, heart and soul, to the work of the invention of the Logarithms. This work possesses an "independent value as affording evidence of the exact state of his Algebraic knowledge at the time when Logarithms were invented. But even if he had written nothing and invented only the calculus of Logarithms, Napier's would have been an abiding name in the history of Mathematics.

In an unpublished inaugural address on "John Napier, Baron of Marchiston," Prof. Chrystal said "whether we consider the great originality of the idea, the difficulty of carrying it into effect in the state in which Algebraical analysis then was, or the immense practical and theoretical value of the invention, we shall have little difficulty in claiming for Napier the honour of a discoverer un-surpassed in brilliancy in the history of Mathematics The German poet Novalis said "what Logarithms, are to the Mathematics that Mathematics are to other Sciences." And we may agree with Dr. Glaishert hat "with the exception of the Principia of Newton, there is no mathematical work published in the country which has produced such important consequences or to which so much interest attaches as to Napier's "Descriptio."

Primitive matrices and the primitive degrees of a matrix.

Part I.

BY

C. E. CULLIS.

1. Introductory remarks.

In the present paper all functions are rational and integral in a finite number of algebraic variables $x, y, z, \ldots$, the co-efficients being given algebraic numbers; and all matrices are such that their elements are rational integral functions of those variables. The first four articles are purely descriptive, and contain no complete proofs.

By the *degree of a non-zero function* is meant its total degree in all the variables, *i.e* the greatest value of the sum of the indices of $x, y, z, \ldots$ in the various non-vanishing terms of the function. The *degree of a non-zero matrix*, or of any non-zero row of a matrix, is the highest degree of its various non-vanishing elements. The degree of a non-zero function or matrix cannot be less than 0. We shall sometimes speak of degrees less than 0 and consider that a function vanishes identically, or that a matrix or a row of a matrix has rank 0, *i.e.* vanishes identically, when and only when its degree is less than 0.

The horizontal rows of a matrix $A=[a]_m^n$ are *connected* or *unconnected* according as there does or does not exist a rational integral equation of the form $[h]_m [a]_m^n = 0$ in which the elements of $[h]_m$ do not all vanish identically, the equation being an identity in the variables $x, y, z, \ldots$. Similarly the vertical rows of the matrix $A=[a]_m^n$ are *connected* or *unconnected* according as there does or does not exist a rational integral equation of the form $[a]_m^n \overset{\sqcap}{\underset{\sqcup}{k}}_n = 0$ in which the elements of $\overset{\sqcap}{\underset{\sqcup}{k}}_n$ do not all vanish identically.

A matrix $[a]_n$ consisting of a single horizontal row is connected with the horizontal rows of the matrix $A=[a]_m^n$ when there exists a rational integral equation of the form

$$\omega[a]_n = [h]_m^{\prime} [a]_m^n \tag{1}$$

in which ω does not vanish identically; and similarly a matrix $\overline{\underline{\,a\,}}_m$ consisting of a single vertical row is connected with the vertical rows of the matrix $A=[a]_m^n$ when there exists a rational integral equation of the form

$$\omega\,\overline{\underline{\,a\,}}_m = [a]_m^n\,\overline{\underline{\,k\,}}_n \tag{2}$$

in which ω does not vanish identically; the equations (1) and (2) being identities in the variables $x, y, z, \ldots$.

Ex. If $[c]_n = [c_1\ c_2 \ldots c_n]$ is connected with the horizontal rows of the undegenerate matrix $[a]_r^n$ whose rank is r, and if this connection is expressed in the two forms

$$\omega\,[c]_n = [\phi_1\phi_2\ldots\phi_r]\,[a]_r^n, \qquad \omega'[c]_n = [\psi_1\psi_2\ldots\psi_r]\,[a]_r^n,$$

where ω and ω' do not vanish identically, we must have

$$\omega'\,[\phi_1\phi_2\ldots\phi_r] = \omega\,[\psi_1\psi_2\ldots\psi_r].$$

Therefore ψ_i vanishes identically when and only when ϕ_i vanishes identically, i being any one of the integers $1, 2, \ldots r$.

If $A=[a]_m^n$ is any matrix of rank r whose elements are rational integral functions of $x, y, z, \ldots$, and if i is any one of the integers $1, 2, \ldots r$, the H.C.F. of all minor determinants of A of order i is denoted by D_i, and we call

$$D_1, D_2, D_3, \ldots D_r$$

the *maximum factors* of A of orders $1, 2, 3, \ldots r$. To these functions we add D_0 defined by the equations $D_0=1$. The function D_i is necessarily divisible by D_{i-1}, and if $E_i = \dfrac{D_i}{D_{i-1}}$, we call

$$E_1, E_2, E_3, \ldots E_r$$

the *potent factors* of A of orders $1, 2, 3, \ldots r$. Each of these functions after the first is divisible by the preceding function.

The irreducible factors of D_i and E_i are all included amongst the irreducible factors of D_r, which are the same as the irreducible factors of E_r; and the irreducible factors of D_r or E_r are called the *irreducible divisors* of A. If D_i is expressed as a product of powers of irreducible divisors of A, the factors of D_i are called the *maximum divisors* of A of order i; and if E_i is expressed as a product of powers of irreducible divisors of A, the factors of E_i are called the *potent divisors* of A of order i

The last three sets of quantities all have reference to some (usually the smallest possible) domain of rationality Ω in which all the elements of A lie, an irreducible function being one which lies in Ω and is irreducible in Ω. When Ω is not specified, it will be understood to be unrestricted, and in that case the irreducible divisors of A are its *irresoluble divisors.*

The matrix A is called *impotent* when it has no potent divisors. This is the case when and only when E_r is a non-vanishing constant which we can take to be 1; also when and only when D_r is a non-vanishing constant which we can take to be 1; and then all the D's and E's are non-vanishing constants, and we can write

$$D_1 = D_2 = \ldots = D_r = 1, \quad E_1 = E_2 = \ldots = E_r = 1$$

A rational integral equation (*i.e.* identity in $x, y, z, \ldots$) of the form

$$[h]_m^p \, [a]_p^q \, [k]_q^n = [b]_m^n \tag{3}$$

in which $[h]_m^p$ and $[k]_q^n$ are undegenerate and impotent matrices of ranks p and q is called an *equipotent transformation* converting the matrix $A = [a]_p^q$ into the matrix $B = [b]_m^n$. It is so called because it leaves the rank and the potent divisors of A unchanged.

A rational integral equation (*i.e.* identity in $x, y, z, \ldots$) of the form (3) in which $[h]_m^p$ and $[k]_q^n$ are undegenerate matrices of ranks p and q whose elements are all constants is called an *equigradent transformation* converting the matrix $A = [a]_p^q$ into the matrix $B = [b]_m^n$. It is so called because it leaves not only the rank and the potent divisors of A, but also the primitive degrees and the non-zero minimum degrees of connection of A unaltered.

In this paper the properties of an equipotent transformation mentioned above are regarded as known.

2. Mutually equivalent matrices.

Two matrices $A = [a]_r^n$ and $B = [b]_s^n$ whose vertical orders are equal are *horizontally equivalent* to one another when every horizontal row of A is connected with the horizontal row of B and every horizontal row of B is connected with the horizontal rows of A. When this is so, there must exist equations of the forms

$$ {}^1[\omega]_r\ [a]_r^n = [\phi]_r^s\ [b]_s^n\ , \quad {}^1[\omega']_s\ [b]_s^n = [\phi']_s^r\ [a]_r^n\ , \tag{1} $$

where ${}^1[\omega]_r$ and ${}^1[\omega']_s$ are undegenerate quasi-scalar matrices, and also equations of the forms

$$ \omega\,[a]_r^n = [\psi]_r^s\ [b]_s^n\ , \quad \omega'[b]_s^n = [\psi']_s^r\ [a]_r^n\ , \tag{2} $$

where ω and ω' do not vanish identically; the equations being identities in the variables $x, y, z, \ldots\ldots$. We can derive equations of the form (2) from (1) by prefixing the conjugate reciprocals of ${}^1[\omega]_r$ and ${}^1[\omega']_s$; and we can derive equations of the form (1) from (2) by cancelling on both sides factors of the successive horizontal rows of $[\psi]_r^s$ which are also factors of ω, or factors of the successive horizontal rows of $[\psi']_s^r$ which are also factors of ω'. From (1) or (2) we see that the rank of A cannot exceed the rank of B, and that the rank of B cannot exceed the rank of A. Therefore when A and B are horizontally equivalent, they must have equal ranks. The most useful necessary and sufficient conditions that A and B shall be horizontally equivalent are as follows:

If the matrices $A = [a]_r^n$ *and* $B = [b]_s^n$ *have the same rank, they are horizontally equivalent when and only when every horizontal row of B is connected with the horizontal rows of A, i. e. when and only when there exists an identity of in x, y, z, of one of the forms*

$$ \omega\,[b]_s^n = [\phi]_s^r\ [a]_r^n\ , \quad {}^1[\omega]_s\ [b]_s^n = [\phi]_s^r\ [a]_r^n\ , \tag{3} $$

where $\omega \neq 0$, *and* ${}^1[\omega]_s$ *is an undegenerate quasi-scalar matrix.*

If $s \not< r$, *then in all cases the two matrices* $A = [a]_r^n$ *and* $B = [b]_s^n$ *are horizontally equivalent to one another when and only when there exists an identity in x, y, z,...of one of the forms (3) in which* $[\phi]_s^r$ *is an undegenerate matrix of rank r.*

Two matrices $A = [a]_m^r$ and $B = [b]_m^s$ whose horizontal orders are equal are *vertically equivalent* to one another when every vertical row of A is connected with the vertical rows of B and every vertical row of B is connected with the vertical rows of A. When these two matrices A and B are vertically equivalent, they must have equal ranks. The most useful necessary and sufficient conditions for their vertical equivalence are as follows:

If the matrices $A = [a]_m^r$ *and* $B = [b]_m^s$ *have the same rank, they are vertically equivalent when and only when every vertical row of B is connected with the vertical rows of A, i.e. when and only when there exists an identity in x, y, z,... of one of the forms*

$$\omega\,[b]_m^s = [a]_m^r\,[\phi]_r^s\,, \qquad [b]_m^s\,{}^1[\omega]_s = [a]_m^r\,[\phi]_r^s\,, \qquad (4)$$

where $\omega \neq 0$, *and* ${}^1[\omega]_s$ *is an undegenerate quasi-scalar matrix.*

If $s \nless r$, *then in all cases the two matrices* $A = [a]_m^r$ *and* $B = [b]_m^s$ *are vertically equivalent to one another when and only when there exists an identity in x, y, z,.. of one of the forms (4) in which* $[\phi]_r^s$ *is an undegenerate matrix of rank r.*

When two *similar undegenerate* matrices are either horizontally or vertically equivalent to one another, we can omit the qualifying term 'horizontally' or 'vertically', and speak of them as being simply *mutually equivalent.* There is no ambiguity in doing so; for there could only be ambiguity when the matrices are square, as well as undegenerate and similar, and in that case they have both horizontal and vertical equivalence.

Thus if $A = [a]_r^n$ and $B = [b]_r^n$ are two similar undegenerate matrices of rank r, they are mutually equivalent when and only when there exists an equation of the form

$$\omega\,[b]_r^n = [\phi]_r^r\;[a]_r^n \qquad (5)$$

in which $\omega \neq 0$; and if $A = [a]_m^r$ and $B = [b]_m^r$ are two similar undegenerate matrices of rank r, they are mutually equivalent when and only when there exists an equation of the form

$$\omega\,[b]_m^r = [a]_m^r\,[\phi]_r^r \qquad (6)$$

in which $\omega \neq 0$. In both the equations (5) and (6) the square matrix $[\phi]_r^r$ is necessarily undegenerate.

Ex. If the two matrices $A=[a]_r^n$ and $B=[b]_s^n$ are horizontally equivalent to one another, a one-rowed matrix $[c_1\ c_2 \ldots c_n]$ is connected with the horizontal rows of A when and only when it is connected with the horizontal rows of B.

3. Primitive one-rowed matrices.

A matrix $[a_1\ a_2 \dots a_n]$ consisting of a single horizontal row is *primitive* when it is undegenerate and impotent, *i.e.* when it is a non-zero matrix whose elements have no common factor other than a non-vanishing constant. When it is a non-zero matrix but not primitive, it can be expressed in the form

$$[a_1\ a_2 \dots a_n] = \omega\ [h_1\ h_2 \dots h_n],$$

where ω is the H. C, F. of all its elements, and then $[h_1\ h_2 \dots h_n]$ is primitive.

Primitive matrices consisting of a single vertical row are defined in a similar way.

The following theorem will be useful in later articles.

If $A = [a]_r^n$ is a matrix whose horizontal rows are unconnected, i.e. an undegenerate matrix of rank r, and if $[h_1\ h_2 \dots h_n]$ is a primitive one-rowed matrix connected with the horizontal rows of A, then there exists a rational integral identity in the variables $x, y, z, \dots$ of the form

$$E_r\ [h_1\ h_2 \dots h_n] = [f]_1^r\ \ [a]_r^n\ , \tag{1}$$

where E_r is the potent factor of A of order r.

There certainly exists a rational integral identity of the form

$$\omega\ [h_1\ h_2 \dots h_n] = [u]_1^r\ \ [a]_r^n \tag{2}$$

in which $\omega \neq 0$ and $[u]_1^r$ is primitive. The function ω must then be a factor of E_r, and there exists a rational integral function ω' such that $\omega\,\omega' = E_r$. Multiplying both sides of (2) by ω', we obtain an identity of the form (I).

4. Primitive matrices in general.

Let the degrees of the successive horizontal rows of the matrix $A = [a]_r^n$ be $\epsilon_1, \epsilon_2, \dots \epsilon_r$. Then A will be said to be an *underanged horizontally primitive matrix* when the following conditions are satisfied :

(1). The horizontal rows of A are unconnected.

(2). There is no non-zero one rowed matrix connected with the horizontal rows of A which has degree less than ϵ_1.

(3). There is no non-zero one-rowed matrix connected with the horizontal rows of A but not with the horizontal rows of $[a]_{i-1}^n$

which has degree less than ϵ_i, this being true for all the values $2, 3, \ldots r$, of i.

These conditions can clearly only be satisfied when the integers $\epsilon_1, \epsilon_2, \ldots \epsilon_r$ (which are not necessarily all different, and not necessarily all different from 0) are arranged in ascending order of magnitude. They can be replaced by the following conditions, which are equivalent to them :

(1′) $[a_{11}\ a_{12} \ldots a_{1n}]$ is a non-zero one-rowed matrix of the lowest possible degree connected with the horizontal rows of A.

(2′) $[a_{i1}\ a_{i2} \ldots a_{in}]$ is a non-zero one-rowed matrix of the lowest possible degree connected with the horizontal rows of A but not with the horizontal rows of $[a]_{i-1}^{n}$, this being true for all the values 2, 3,...r of i.

The necessary condition that the horizontal rows of A are unconnected is included in (1′) and (2′).

If A is not an underanged horizontally primitive matrix but can be converted into one by derangements of its horizontal rows, it will be called a *deranged horizontally primitive matrix*. In general A is a *horizontally primitive matrix* when it is either an underanged horizontally primitive matrix or a horizontal derangement of such a matrix. It is only possible for a matrix to be horizontally primitive when its horizontal rows are unconnected, and are all primitive.

If $A = [a]_{m}^{n}$ is any matrix of rank r, any horizontally primitive matrix which is horizontally equivalent to A, must be an undegenerate matrix of rank r whose horizontal rows are unconnected ; and $B = [b]_{r}^{n}$ will be an *underanged horizontally primitive matrix horizontally equivalent to A* when and only when the following conditions are satisfied :

(1″) $[b_{11}\ b_{12} \ldots b_{1n}]$ is a non-zero one-rowed matrix of the lowest possible degree connected with the horizontal rows of A.

(2″). $[b_{i1}\ b_{i2} \ldots b_{in}]$ is a non-zero one-rowed matrix of the lowest possible degree connected with the horizontal rows of A but not with the horizontal rows of $[b]_{i-1}^{n}$, this being true for all the values 2, 3,...r of i.

For when $B = [b]_{r}^{n}$ is a matrix of rank r horizontally equivalent to A, the non-zero one-rowed matrices connected with the horizontal rows of B are identical with the non-zero one-rowed matrices connected with the horizontal rows of A, and the conditions similar to (1′) and (2′) that B shall be an underanged horizontally primitive matrix lead to (1″) and (2″). Conversely when the conditions (1″) and (2″) are satisfied, $B=[b]_r^n$ is an undegenerate matrix of rank r whose horizontal rows are connected with the horizontal rows of A; it is therefore horizontally equivalent to A; therefore the non-zero one-rowed matrices connected with the horizontal rows of A are identical with the non-zero one-rowed matrices connected with the horizontal rows of B, and it therefore follows from (1″) and (2″) that B is an underanged horizontally primitive matrix.

Of course B is a *horizontally primitive matrix horizontally equivalent to A*, when it is either an underanged horizontally primitive matrix horizontally equivalent to A, or is a horizontal derangement of such a matrix.

If $C=[c]_r^n$ is any undegenerate matrix of rank r horizontally equivalent to A, in particular if C is any undegenerate horizontal minor of A of reduced order r, then the horizontally primitive matrices horizontally equivalent to A are the same as the horizontally primitive matrices horizontally equivalent to C.

We can in a similar way define a *vertically primitive matrix* and those vertically primitive matrices which are vertically equivalent to a given matrix. It is only possible for a matrix to be vertically primitive when its vertical rows are unconnected, and are all primitive. A vertically primitive matrix is *underanged* when its vertical rows are so arranged that their degrees are in ascending order of magnitude.

When a matrix whose orders are given is either horizontally or vertically primitive, we can omit the qualifying term 'horizontally' or 'vertically', and speak of it simply as being *primitive.* There is no ambiguity in doing so; for there could only be ambiguity if the matrix were square, and in that case (see Ex. iv of Art 6.) it is horizontally primitive when and only when it is vertically primitive.

Ex. If $A=[a]_r^n$ is an undegenerate and impotent rational integral functional matrix of rank r whose elements are all homogeneous and linear in the variables, it must be horizontally primitive.

In this case we can take the potent factor of A of order r to be 1, and if $[h_1\,h_2 \ldots h_n]$ is any primitive one-rowed matrix connected with the horizontal rows of A, then by Art. 3 there must exist a rational integral equation of the form

$$[h_1 h_2 \dots h_n]=[f]_1^r \; [a]_r^n .$$

Since the right hand side vanishes when the variables all vanish, no one of the elements of $[h]_n$ can be a non-vanishing constant, and $[h]_n$ cannot have degree 0. Thus there is no one-rowed matrix of degree 0 connected with the horizontal rows of A, and therefore A must be primitive.

5. Some tests for the primitivity of a matrix.

We consider here certain tests for the horizontal primitivity of a matrix $A=[a]_r^n$ whose horizontal rows are unconnected. There are similar tests for the vertical primitivity of a matrix whose vertical rows are unconnected.

First let the degrees of the 1st, 2nd, ..rth horizontal rows of A be $\epsilon_1, \epsilon_2, \dots \epsilon_r$, where these are positive integers (not necessarily all different, and not necessarily all different from 0) arranged in ascending order of magnitude, so that if A is horizontally primitive, it must be an *underanged* horizontally primitive matrix. In this case A is horizontally primitive when and only when any one of the following three equivalent sets of conditions is satisfied, where in each case ϕ_0 is some rational integral function which does not vanish identically, and all the values 1, 2, ..r can be ascribed to ι.

(A). *In every rational integral identity in $x, y, z,\dots$ of the form*

$$[\phi_1 \; \phi_2 \dots \phi_r] \; [a]_r^n \; = \; \phi_0 \; [h_1 \; h_2 \dots h_n] \tag{1}$$

in which $\phi_\iota, \phi_{\iota+1}, \dots \phi_r$ do not all vanish identically, the matrix $[h]_n$ has degree not less than ϵ_ι.

(B). *In every rational integral identity in $x, y, z,\dots$ of the form (1) in which ϕ_ι does not vanish identically, the matrix $[h]_n$ has degree not less than ϵ_ι:*

(C). *In every rational integral identity in $x, y, z,\dots$ of the form*

$$[\phi_1 \; \phi_2 \dots \phi_\iota] \; [a]_\iota^n \; = \; \phi_0 \; [h_1 \; h_2 \dots h_n] \tag{2}$$

in which ϕ_ι does not vanish identically, the matrix $[h]_n$ has degree not less than ϵ_ι.

The test (A) needs no proof, for by the example in Art. 1 it is merely an analytical formulation of the definition of underanged primitivity given in Art. 4.

If the matrix A is horizontally primitive, then the conditions (A) must be satisfied, and it follows that the conditions (B) and (C) must be satisfied. Thus (B) and (C) are necessary conditions for the horizontal primitivity of A, and it only remains to show that they are also sufficient conditions.

If the conditions (B) are satisfied, then in every identity of the form (1) in which $\phi_i, \phi_{i+1}, \ldots \phi_r$ do not all vanish identically, the degree of $[h]_n$ is not less than some one of the integers $\epsilon_i, \epsilon_{i+1}, \ldots \epsilon_r$, and is therefore not less than ϵ_i; consequently the conditions (A) are satisfied, and therefore A is horizontally primitive.

If the conditions (C) are satisfied for all the values $1, 2, \ldots r$ of i, then in every identity of the form (1) in which $\phi_i, \phi_{i+1}, \ldots \phi_r$ do not all vanish identically, the degree of $[h]_n$ is not less than ϵ_κ, where ϕ_κ is the last of the functions $\phi_i, \phi_{i+1}, \ldots \phi_r$ which does not vanish identically. Since $\kappa \not< i$ and $\epsilon_\kappa \not< \epsilon_i$, it follows that the degree of $[h]_n$ is not less than ϵ_i. Accordingly the conditions (A) are satisfied, and therefore the matrix A is horizontally primitive.

Thus each of the sets of conditions (B) and (C) is sufficient as well as necessary for the primitivity of A; and this proves the tests (B) and (C).

If no horizontal row of A vanishes, we must consider that no one of the integers $\epsilon_1, \epsilon_2, \ldots \epsilon_r$ is less than O. In this case each of the sets of conditions (A), (B), (C) includes the necessary condition that the horizontal rows of A are unconnected, and this condition is superfluous; for when any one of the sets of conditions (A), (B), (C) is satisfied, there cannot exist any relation of the form $[\phi_1\ \phi_2 \ldots \phi_r]\ [a]_r^n = 0$ in which $\phi_1, \phi_2, \ldots \phi_r$ do not all vanish identically.

If we retain the condition that the horizontal rows of A are unconnected, then in applying any one of the foregoing tests there will clearly be no loss of generality if we restrict $[h]_n$ to be a primitive one-rowed matrix, and then by Art. 3 we can always take ϕ_κ to be E_r, the potent factor of A of order r.

Next let the degrees of the 1st, 2nd,...*rth* horizontal rows of A be $s_1, s_2, \ldots s_r$, where these are any positive integers arranged in any order. In this case we have the following two results, in each of which ϕ_κ is some rational integral function which does not vanish identically, and all the values $1, 2, \ldots r$ are to be ascribed to i.

(A′). *The matrix* $A=[a]_r^n$ *is an underanged horizontally primitive matrix when and only when in every rational integral identity in* $x, y, z,\ldots$ *of the fosm*

$$[\phi_1\ \phi_2\ldots\phi_r]=\phi_0\ [h_1\ h_2\ldots h_n] \qquad (1)$$

in which $\phi_i, \phi_{i+1},\ldots\phi_r$ *do not all vanish identically, the matrix* $[h]_n$ *has degree not less than* s_i.

(B′). *The matrix* $A=[a]_r^n$ *is horizontally primitive when and only when in every rational integral identity in* $x, y, z,\ldots$ *of the form (1) in which* ϕ_i *does not vanish identically, the matriv* $[h]_n$ *has degree not less than* s_i.

The test (A′) needs no proof, for it is merely an analytical formulation of the definition of underanged primitivity given in Art. 4. It will be observed that the conditions (A′) can only be satisfied when $s_1, s_2,\ldots s_r$ are arranged in ascending order of magnitude.

The test (B′) follows immediately from the test (B).

When the horizontal rows, of A are known to be unconnected, then in applying the tests (A′) and (B′) we can without loss of generality restrict $[h]_n$ to be primitive, and take ϕ_0 to be E_r, the potent factor of A of order r.

Ex. i. If $A=[a]_r^n$ *is a horizontally primitive matrix, then every horizontal minor of A is uudegenerate and primitive; in particular every horizontal row of A is undegenerate and primitive.*

This follows from any one of the tests given above.

Ex. ii. If $A=[a]_r^n$ *is an undegenerate and impotent matrix of rank* r *in which every horizontal row is homogeneous in the variables, then A must be horizontally primitive.*

Let the horizontal rows of A be so arranged that their successive degrees $\epsilon_1, \epsilon_2,\ldots\epsilon_r$ are in ascending order of magnitude. Then since we can take the potent factor of A of order r to be 1, it follows from the test (C) that if A is not horizontally primitive, there would exist for one of the values $1, 2,\ldots r$ of i a rational integral identity in the variables of the form

$$[\phi_1\ \phi_2\ldots\phi_i]\ [a]_i^n = [h]_1^n \qquad (3)$$

in which $\phi_i \neq 0$, and $[h]_1^n$ is a primitive one rowed matrix whose degree is less than ϵ_r. By equating to zero the terms of any degree not less than ϵ_r which occur in the product on the left in (3) we should then obtain an identity of the form

$$[\psi_1\ \psi_2 \dots \psi_r]\,[a]_r^n = 0 \qquad (4)$$

in which $\psi_1, \psi_2 \dots \psi_r$ are homogeneous functions which do not all vanish, their degrees $\tau_1, \tau_2, \dots \tau_r$ being such that

$$\tau_1+\epsilon_1=\tau_2+\epsilon_2=\dots=\tau_r+\epsilon_r.$$

This however is impossible because the horizontal rows of A are unconnected. Consequently A must be primitive.

6. The primitive degrees of a matrix.

The definitions of the primitive degrees of any matrix $A=[a]_m^n$ of rank r whose elements are rational integral functions of the variables $x, y, z, \dots$ depend on the following two theorems.

Theorem I. *If $B=[b]_r^n$ and $B'=[b']_r^n$ are any two underanged horizontally primitive matrices which are horizontally equivalent to A, and if the degrees of the successive horizontal rows of B and B' are respectively $\epsilon_1, \epsilon_2, \dots \epsilon_r$ and $\epsilon'_1, \epsilon'_2, \dots \epsilon'_r$, then:*

(1). $\epsilon'_1=\epsilon_1,\ \epsilon'_2=\epsilon_2, \dots \epsilon'_r=\epsilon_r$.

(2). $[b]_i^n$ *and* $[b']_i^n$ *are mutually equivalent whenever* $\epsilon_{i+1}\neq\epsilon_i$.

Theorem II. *If $B=[b]_m^r$ and $B'=[b']_m^r$ are any two underanged vertically primitive matrices which are vertically equivalent to A, and if the degrees of the successive vertical rows of B and B' are respectively $\eta_1, \eta_2, \dots \eta_r$ and $\eta'_1, \eta'_2, \dots \eta'_r$, then:*

(1) $\eta_1'=\eta_1,\ \eta'_2=\eta_2, \dots \eta'_r=\eta_r$.

(2). $[b]_m^i$ *and* $[b']_m^i$ *are mutually equivalent whenever* $\eta_{i+1}\neq\eta_i$.

Here i is any one of the integers $1, 2, \dots r$; and each of the series $(\epsilon_1, \epsilon_2, \dots \epsilon_r)$, $(\epsilon'_1, \epsilon'_2, \dots \epsilon'_r)$, $(\eta_1, \eta_2, \dots \eta_r)$, $(\eta'_1\ \eta'_2 \dots \eta'_r)$ consists of r positive integers arranged in ascending order of magnitude The r integers in each series are however not necessarily all different, and some or all of them may have the value 0.

We will prove the second theorem only, the proof of the first theorem being similar; and we will deduce Theorem II from the results obtained in the three special cases which follow, where for the sake of brevity we use the notations.

$$b_i=\begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix}, \quad b'_i=\begin{bmatrix} b'_{1i} \\ b'_{2i} \\ \vdots \\ b'_{mi} \end{bmatrix}, \quad c_i=\begin{bmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{mi} \end{bmatrix}, \quad \dots\,.$$

CASE I. *Suppose that $\eta_1, \eta_2 \dots \eta_r$ are all different.*

In this case the following statement is true for the values 1, 2,... $r-1$ of i.

Every non-zero one-rowed matrix connected with the vertical rows of A but not with the vertical rows of $[b]^i_m$ has degree greater than η_i. (1)

Since b'_1 is defined in exactly the same way as b_1, we must have $\eta'_1=\eta_1$; and it follows from (1) that b'_1 is connected with b_1, so that $[b]^1_m$ and $[b']^1_m$ are mutually equivalent.

Therefore b'_2 can be defined in exactly the same way as b_2, and we must have $\eta'_2=\eta_2$. It then follows from (1) that b'_2 is connected with the vertical rows of $[b]^2_m$, so that $[b]^2_m$ and $[b']^2_m$ are mutually equivalent.

Continuing in this way we see that $\eta'_i=\eta_i$, and that $[b]^i_m$ and $[b']^i_m$ are mutually equivalent, this being true for all the values 1, 2,...r of i.

Accordingly Theorem II is true in Case I

CASE II. *Suppose that* $\eta_1=\eta_2=\dots=\eta_s=\eta$, *and that* $\eta_{s+1}>\eta$.

It is assumed that $s<r$, but it will be shown that the results obtained in this case remain true when $s=r$, the condition $\eta_{s+1}>\eta$ being then omitted.

From the definition of an underanged vertically primitive matrix we see that the following two statements are true.

No non-zero one-rowed matrix connected with the vertical rows of A has degree less than η. (2)

Every non-zero one-rowed matrix connected with the vertical rows of A but not with the vertical rows of $[b]^s_m$ has degree greater than η. (2′)

Since b'_1 is defined in exactly the same way as b_1, we must have $\eta'_1=\eta_1=\eta$; and it follows from (2′) that b'_1 is connected with the vertical rows of $[b]^{s}_{m}$. We may suppose without loss of generality that there exists a rational integral equation of the form

$$\omega b'_1 = [b]^{s}_{m} \overline{\underline{\mu}}_{s}$$

in which ω and μ_s do not vanish identically (this being necessarily the case when a suitably chosen vertical row of $[b]^{s}_{m}$ is brought to the final position), so that $[b', b]^{1, s-1}_{m}$ and $[b]^{s}_{m}$ are mutually equivalent.

Then because every vertical row of $[b]^{s-1}_{m}$ is a non-zero matrix of degree η connected with the vertical rows of A but not with $[b']^{1}_{m}$ we see that $\eta'_2 \ngtr \eta$. Therefore by (2) we must have $\eta'_2 = \eta$; and it follows from (2′) that b'_2 is connected with the vertical rows of $[b]^{s}_{m}$ and therefore with those of $[b', b]^{1, s-1}_{m}$. Since b'_2 is not connected with b'_1, we may suppose without loss of generality that there exists a rational integral equation of the form

$$\omega b'_2 = [b', b]^{1, s-1}_{m} \overline{\underline{\begin{matrix}\lambda \\ \mu\end{matrix}}}_{1, s-1}$$

in which ω and μ_{s-1} do not vanish identically (this being necessarily the case when the vertical rows of $[b]^{s}_{m}$ are suitably arranged), so that $[b', b]^{2, s-2}_{m}$ and $[b', b]^{1, s-1}_{m}$ or $[b]^{s}_{m}$ are mutually equivalent.

Then because every vertical row of $[b]^{s-2}_{m}$ is a non-zero matrix of degree η connected with the vertical rows of A but not with the vertical rows of $[b']^{2}_{m}$ we see that $\eta'_3 \ngtr \eta$. Therefore by (2) we must have $\eta'_3 = \eta$; and it follows from (2′) that b'_3 is connected with the vertical rows of $[b]^{s}_{m}$, and therefore with those of $[b', b]_m^{2,\, 2}$. Since b'_s is not connected with the vertical rows of $[b', b]_m^{2}$, we may suppose without loss of generality that there exists a rational integral equation of the form

$$\omega\, b'_s = [b', b]_m^{2,\, s-2} \begin{array}{c} \overline{\lambda} \\ \mu \\ \overline{}\, 2,\, s-2 \end{array}$$

in which ω and μ_{s-2} do not vanish identically, so that $[b', b]_m^{3,\, s-3}$ and $[b', b]_m^{2,\, s-2}$ or $[b]_m^{s}$ are mutually equivalent.

Continuing in this way we see that:

In Case II we must have $\eta_1' = \eta_2' = \dots = \eta_s' = \eta$; *and* $[b]_m^{s}$ *and* $[b']_m^{s}$ *are mutually equivalent.*

If $s = r$, we must replace (2′) by the statement that every non-zero one-rowed matrix connected with the vertical rows of A is connected with the vertical rows of $[b]_m^{s}$, and we then obtain the same results,

If $s < r$, the matrix b'_{s+1} can be defined in exactly the same way as b_{s+1}, and we must have $\eta'_{s+1} = \eta_{s+1} > \eta$.

CASE III. *Suppose that* $\eta_u < \eta$, $\eta_{u+1} = \eta_{u+2} = \dots = \eta_{u+s} = \eta$, $\eta_{u+s+1} > \eta$; *and suppose that it has been proved that* $\eta_1' = \eta_1$, $\eta_2' = \eta_2$, $\dots$ $\eta_u' = \eta_u$, *and that* $[b]_m^{u}$ *and* $[b']_m^{u}$ *are mutually equivalent.*

It is assumed that $u + 1 > 1$ and $u + s < r$, but it will be shown that the results obtained remain true when $u + s = r$ and the condition $\eta_{u+s+1} > \eta$ is omitted,

We will write $[b]_m^{r} = [b,\ c,\ d]_m^{u,\, s,\, r-u-s}$.

Then in this case the following statements are true.

No non-zero one-rowed matrix connected with the vertical rows of A, but not with the vertical rows of the mutually equivalent matrices $[b]_m^{u}$ *and* $[b']_m^{u}$, *can have degree less than* η. (3)

Every non-zero one-rowed matrix connected with the vertical rows of A, but not with the vertical rows of the mutually equivalent matrices $[b, c]_m^{u,\, s}$ *and* $[b', c]_m^{u,\, s}$ *has degeee greater than* η. (3′)

Since $b'_{\kappa+1}$ can be defined in exactly the same way as $b_{\kappa+1}$, we must have $\eta'_{\kappa+1} = \eta_{\kappa-1} = \eta$: and it follows from (3′) that $b'_{\kappa+1}$ is connected with the vertical rows of $[b', c]^{u,s}_m$. Because it is not connected with the vertical rows of $[b']^{u}_m$, we may suppose without loss of generality that there exists a rational integral equation of the form

$$\omega\, b'_{\kappa+1} = [b', c]^{u,s}_m \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{u,s}$$

in which ω and μ_s do not vanish identically, so that $[b', c]^{u+1,s-1}_m$ and $[b', c]^{u,s}_m$ or $[b, c]^{u,s}_m$ are mutually equivalent.

Then because every vertical row of $[c]^{\kappa-1}_m$ is a non-zero matrix of degree η connected with the vertical rows of A but not with the vertical rows of $[b']^{u+1}_m$, we see that $\eta'_{\kappa+2} \not> \eta$. Therefore by (3) we must have $\eta'_{\kappa-2} = \eta$, and it follows from (3′) that $b'_{\kappa+2}$ is connected with the vertical rows of $[b, c]^{u,s}_m$ and therefore with those of $[b', c]^{u+1,s-1}_m$, but not with those of $[b']^{u+1}_m$. Accordingly we may suppose without loss of generality that there exists a rational integral equation of the form

$$\omega\, b'_{\kappa+2} = [b', c]^{u+1,s-1}_m \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{u+1,s-1}$$

in which ω and μ_{s-1} do net vanish identically, so that

$(b', c]^{u+2,s-2}_m$ and $[b', c]^{u+1,s-1}_m$ or $[b, c]^{u,s}_m$ are mutually equivalent.

Then because every vertical row of $[c]^{s-2}_m$ is a non-zero matrix of degree η connected with the vertical rows of A but not with the vertical rows of $[b']^{u+2}_m$, we see that $\eta'_{\kappa+3} > \eta$. Therefore by (3) we must have $\eta'_{\kappa+3} = \eta$; and it follows from (3′) that $b'_{\kappa+3}$ is connected with the vertical rows of $[b, c]^{u,s}_m$, and therefore with those of $[b', c]^{u+2,s-2}_m$, but not with those of $[b']^{u+2}_m$. Accordingly we may assume without loss of generality that there exists a rational integral equation of the form

$$\omega\, b'_{u+s} = [b', c]_m^{u+2,\, s-2} \begin{bmatrix} \lambda \\ \mu_{u+2,\, s-2} \end{bmatrix}$$

in which ω and μ_{s-2} do not vanish identically, so that $[b', c]_m^{u+3,\, s-3}$ and $[b', c]_m^{u+2,\, s-2}$ or $[b, c]_m^{u,\, s}$ are mutually equivalent.

Continuing in this way we see that:

In Case III we must have $\eta'_{u+1} = \eta'_{u+2} = \dots = \eta'_{u+s} = \eta$; *and* $[b']_m^{u+s}$ *and* $[b, c]_m^{u,\, s}$ *are mutually equivalent, i.e.,* $[b]_m^{u+s}$ and $[b']_m^{u+s}$ *are mutually equivalent.*

If $u+s = r$, we must replace (3′) by the statement that every non-zero one-rowed matrix connected with the vertical rows of A is connected with the vertical rows of $[b]_m^r$, and we then obtain the same results.

If $u + s < r$, the matrix b'_{u+s+1} can be defined in exactly the same way as b_{u+s+1}, and we must have $\eta'_{u+s+1} = \eta_{u+s+1} > \eta$

From the results obtained in Cases II and III it follows by induction that Theorem II is completely true; and in a similar way we can show that Theorem I is completely true.

Now as before let $A = [a]_m^n$ be any given matrix of rank r.

If $B = [b]_r^n$ is any underanged horizontally primitive matrix whatever which is horizontally equivalent to A, and if $\epsilon_1, \epsilon_2, \dots \epsilon_r$ are the degrees of the 1st, 2nd, ... rth horizontal rows of B, it follows from Theorem I that the integers $\epsilon_1, \epsilon_2, \dots \epsilon_r$ are always the same. We call them the *horizontal primitive degrees of A.* They are r positive integers arranged in ascending order of magnitude, but they are not necessarily all different, and some or all of them may have the value 0.

Again if $B = [b]_m^r$ is any underanged vertically primitive matrix whatever which is vertically equivalent to A, and if $\eta_1, \eta_2, \dots \eta_r$ are the degrees of the 1st, 2nd, ... rth vertical rows of B, it follows from Theorem II that the integers $\eta_1, \eta_2, \dots \eta_r$ are always the same. We call them the *vertical primitive degrees of A.* They are r positive integers arranged in ascending order of magnitude, but they are not necessarily all different, and some or all of them may have the value 0.

The characteristic properties of these primitive degrees are stated more precisely in the next two theorems.

Theorem III. *If $\epsilon_1, \epsilon_2, \dots \epsilon_r$ are the horizontal primitive degrees of the matrix $A = [a]_m^n$ arranged in ascending order of magnitude, and if $B = [b]_r^n$ is any undegenerate matrix horizontally equivalent to A, then B is an underanged horizontally primitive matrix when and only when the degrees of its successive horizontal rows are $\epsilon_1, \epsilon_2, \dots \epsilon_r$.*

Theorem IV. *If $\eta_1, \eta_2, \dots \eta_r$ are the vertical primitive degrees of the matrix $A = [a]_m^n$ arranged in ascending order of magnitude, and if $B = [b]_m^r$ is any undegenerate matrix vertically equivalent to A, then B is an underanged vertically primitive matrix when and only when the degrees of its successive vertical rows are $\eta_1, \eta_2, \dots \eta_r$.*

We prove Theorem III only, the proof of Theorem IV being similar.

First suppose that B in Theorem III is an underanged horizontally primitive matrix. Then, as shown above, the degrees of its successive horizontal rows are $\epsilon_1, \epsilon_2, \dots \epsilon_r$. Next suppose that the degrees of the successive horizontal rows of B are $\epsilon_1, \epsilon_2, \dots \epsilon_r$. Then if B is not an underanged horizontally primitive matrix, it follows from the definition of underanged primitivity that we can determine an underanged horizontally primitive matrix horizontally equivalent to B, and therefore horizontally equivalent to A, in which the degrees of the successive horizontal rows are not all the same as $\epsilon_1, \epsilon_2, \dots \epsilon_r$, and by Theorem I this is impossible. Therefore B must be an underanged horizontally primitive matrix.

Ex. i. In Theorem III the matrix B is horizontally primitive when and only when the degrees of its r horizontal rows are the integers $\epsilon_1, \epsilon_2, \dots \epsilon_r$ arranged in some order.

Ex. ii. Let $A = [a]_r^n$ be a given matrix whose horizontal rows are unconnected; let $B = [b]_r^n$ be an undegenerate matrix horizontally equivalent to A; and let degrees of the r horizontal rows of A and B, arranged in each case in ascending, order of magnitude, be respectively $\epsilon_1, \epsilon_2, \dots \epsilon_r$ and $\epsilon_1', \epsilon_2', \dots \epsilon_r'$. Then A is horizontally primitive when and only when *all* the inequalities

$$\epsilon_1 \not> \epsilon_1', \ \epsilon_2 \not> \epsilon_2', \ \dots \ \epsilon_r \not> \epsilon_r'$$

are always true for all such matrices as B. Thus A is horizontally primitive if, when we compare it with all horizontally equivalent similar undegenerate matrices, each of the successive integers $\epsilon_1, \epsilon_2, \dots \epsilon_r$ has its least possible value.

Ex. iii. If $\mathrm{A} = [a]_m^n$ is a matrix of rank r whose vertical primitive degrees are

u equal to η_1, v equal to η_2, w equal to $\eta_3, \dots$.

where $\eta_1, \eta_2, \eta_3, \dots$ are unequal positive integers arranged in ascending order of magnitude, and where $u + v + w + \dots = r$: and if

$$[b]_m^r = [b, c, d, \dots]_m^{u,\, v,\, w,} \quad , \quad [b']_m^r = [b', c', d' \dots]_m^{u,\, v,\, w,}$$

are two underanged vertically primitive matrices vertically equivalent to A; then

$$[b]_m^u, [b']_m^u ; \ [b, c]_m^{u,\, v}, \ [b', c']_m^{u,\, v} ; [b, c, d]_m^{u,\, v,\, w}, [b', c', d']_m^{u,\, v,\, w} ; \dots$$

are pairs of mutually equivalent undegenerate matrices. This follows from the proof of Theorem II.

Here all vertical rows in the successive pairs of matrices

$$[b]_m^u, \ [b']_m^u ; \ [c]_m^v, \ [c']_m^v ; \ [d]_m^w, \ [d']_m^w ; \dots$$

have degrees $\eta_1, \eta_2, \eta_3, \dots$.

Ex. iv. The primitive degrees of an undegenerate square matrix are all O's.

Let the matrix be $\mathrm{A} = [a]_m^m$, let $\overset{m}{\underset{m}{\overline{\underline{\mathrm{A}}}}}$ be the conjugate reciprocal of A, and let $(a)_m^m = \Delta$. Then we have

$$\Delta\, [1]_m^m = \overset{m}{\underset{m}{\overline{\underline{\mathrm{A}}}}}\, [a]_m^m = [a]_m^m\, \overset{m}{\underset{m}{\overline{\underline{\mathrm{A}}}}} .$$

Therefore $[1]_m^m$ is an underanged primitive matrix both horizontally and vertically equivalent to A.

Ex. v. If $A = [a]_r^n$ is a matrix whose horizontal rows are unconnected, then the vertical primitive degrees of A are all O's.

For we can determine an *undegenerate square matrix* of order r which is vertically equivalent to A. This may be any undegenerate square minor of A of order r.

Ex. vi. The primitive degrees of a matrix whose elements are all constants are all 0's; and if the horizontal rows of such a matrix are unconnected, then the matrix is horizontally primitive.

7. Domain of rationality of an equivalent primitive matrix.

If $A=[a]_m^n$ is any rational integral functional matrix lying in a restricted domain of rationality Ω, we can always determine primitive matrices lying in Ω which are respectively horizontally and vertically equivalent to A. In particular if A is real, we can always determine real primitive matrices which are respectively horizontally and vertically equivalent to A. Since we can certainly form undegenerate horizontal and vertical minors of A which are respectively horizontally and vertically equivalent to A, and which necessarily lie in Ω, it will be sufficient to prove these results for undegenerate matrices, *i.e.*, to prove the following two theorems

If $A=[a]_r^n$ is a matrix lying in Ω whose horizontal rows are unconnected, we can determine an underanged horizontally primitive matrix lying in Ω which is horizontally equivalent to A.

If $A=[a]_m^r$ is a matrix lying in Ω whose vertical rows are unconnected, we can determine an underanged vertically primitive matrix lying in Ω which is vertically equivalent to A.

We will prove the second theorem only, the proof of the first being similar.

Let the r vertical primitive degrees of the matrix $A=[a]_m^r$ be

$$u \text{ equal to } \eta_1, \; v \text{ equal to } \eta_2, \; w \text{ equal } \eta_3, \ldots ,$$

where $\eta_1, \eta_2, \eta_3, \ldots$ are unequal positive integers arranged in ascending order of magnitude, and where $u+v+w+\ldots =r$; and let

$$B=[b]_m^r =[b, c, d, \ldots]_m^{u, v, w, \ldots}$$

be any given underanged vertically primitive matrix vertically equivalent to A. Also let E be the potent factor of A of order r so chosen as to lie in Ω. If B does not lie in Ω, we may suppose that it lies in a domain of rationality Ω' formed from Ω by the adjunction of q, where q is a root of an irreducible equation in Ω of finite degree s.

We will first express $[b]_m^u$ in the form

$$[b]_m^u =[\beta]_m^u +q\,[\beta']_m^u +q^2\,[\beta'']_m^u +\ldots \tag{1}$$

where there are s terms on the right, the matrix factor in each term lying in Ω; and we will show that

$$[\beta, \beta', \beta'', \ldots\ ]_m^{u,\, u,\, u,\, \ldots} \tag{1'}$$

is a matrix of rank u each of whose vertical rows is connected with the vertical rows of $[b]_m^u$. By Art. 3 there exists a rational integral equation of the form

$$\mathrm{E}\, [b]_m^u = [a]_m^r\, [f]_r^u \quad ,$$

and when we use the expression (1) for $[b]_m^u$, express $[f]_r^u$ in a similar form, and equate the co-efficients of q on both sides, we obtain identities in the variables of the forms

$$\mathrm{E}\, [\beta]_m^u = [a]_m^r\, [\phi]_r^u\ ,\ \mathrm{E}\, [\beta']_m^u = [a]_m^r\, [\phi']_r^u\ ,\ \mathrm{E}\, [\beta'']_m^u = [a]_m^r\, [\phi'']_r^u\ , \ldots\ .$$

These show that all the vertical rows of the matrix (1′), *no one of which has degree less than* η_1, are connected with the vertical rows of A, *i.e.*, with the vertical rows of B, and therefore with the vertical rows of $[b]_m^u$. It follows that the rank of the matrix (1′) cannot exceed u. Moreover its rank cannot be less than u, because $[b]_m^u$ is a matrix of rank u whose vertical rows are connected with its vertical rows. Consequently the matrix (1′) has rank u. Now let $[b']_m^u$ be a matrix of rank u formed with u unconnected vertical rows of the matrix (1′). Then $[b]_m^u$ and $[b']_m^u$ are mutually equivalent, $[b']_m^u$ lies in Ω, and every one of its vertical rows must have degree η_1, for no vertical row which is not a row of 0's, can have degree less than η_1.

We will next express $[c]_m^v$ in the form

$$[c]_m^v = [\gamma]_m^v + q[\gamma']_m^u + q^2[\gamma'']_m^v + \ldots \quad , \tag{2}$$

where there are s terms on the right, the matrix factor in each term lying in Ω; and we will show that

$$[b',\ \gamma,\ \gamma',\ \gamma'', \ldots\ ]_m^{u,\, v,\, v,\, v,\, \ldots} \tag{2'}$$

is a matrix of rank $u+v$ each of whose vertical rows is connected with the vertical rows of $[b, c]_m^{u,\, v}$. By Art. 3 there exists a rational integral equation of the form

$$E\,[c]_m^v = [a]_m^r\,[f]_r^v \quad ,$$

and by reasoning similar to that used in the preceding step of the proof it follows that all the vertical rows of the matrix (2′), *no one of which has degree greater than* η_2, are connected with the vertical rows of A, *i.e.*, with the vertical rows of B, and therefore with the vertical rows of $[b,\ c]_m^{u,\ v}$. From this it follows that the rank of the matrix (2′) cannot exceed $u+v$. Moreover its rank cannot be less than $u+v$, because $[b,\ c]_m^{u,\ v}$ is a matrix of rank $u+v$ whose vertical rows are connected with its vertical rows. Consequently the matrix (2′) has rank $u+v$, Now let $[b',\ c']_m^{u,\ v}$ be any matrix of rank $u+v$ formed with the first u vertical rows and v of the following vertical rows of the matrix (2′). Then $[b,\ c]_m^{u,\ v}$ and $[b',\ c']_m^{u,\ v}$ are mutually equivalent. Moreover no vertical row of the latter matrix can have degree less than η_1 or greater than η_2, and no vertical row which is unconnected with the vertical rows of $[b']_m^u$, *i.e.*, with the vertical rows of $[b]_m^u$, can have degree less than η_2. Consequently $[b',\ c']_m^{u,\ v}$ is a matrix lying in Ω in which the first u vertical rows have degree η_1, and the next v vertical rows have degree η_2.

Continuing in this way we finally obtain a matrix

$$B' = [b']_m^r = [b',\ c',\ d',\ \ldots\quad]_m^{u,\ v,\ w,\ldots}$$

of rank r lying in Ω which is vertically equivalent to B, *i.e.*, to A, and in which the first u vertical rows have degree η_1, the next v vertical rows have degree η_2, the next w vertical rows have degree $\eta_3, \ldots$, and by Theorem IV in Art. 6 this matrix B' is an *underanged vertically primitive matrix* which is vertically equivalent to A and lies in Ω.

Reply to Prof. Bryan's Criticism*

By

Syamadas Mukhopadhyay

I

I am glad the subject of a brief note of mine, on the stereoscopic representation of a figure in four dimensions, (*Bulletin C. M. S.* Vol. IV, 1912-13, page 15) has interested Prof. Bryan. He gives us another method of representation. From the very imperfect explanations given by him, it is difficult to form a clear conception of his extra-ordinary pair of stereoscopic pictures. I hope he will impart to us further details.

Apparently his method does not aim at visualizing stereoscopically a four-dimensional figure in all its dimensions, at the same time, as my method does, but only at giving, successively, two three-dimensional aspects of a four-dimensional figure, differing in phase by 90°. If so his method does not go very far.

I thought I had described my method in my note with sufficient clearness. The pair of stereoscopic pictures in my method are not two-dimensional, as is the case with Prof. Bryan's method, but three-dimensional, consisting of a pair of rectilineal figures in space, (constructed of wire or thread), standing on an ordinary stereoscopic pair of plane rectilineal figures as bases. The simple experiment I have suggested illustrates the principles of my method. It gives a solution of the problem of four dimensions, by representing before our eyes four lines standing out mutually, at right angles, or, at any rate, a close approximation to such a solution.

The principles on which true vision of four dimensions may be possible, stereoscopically or otherwise, have been already set fourth by Poincarè. Speaking of complete vision he says. (*Science and Hypothesis*, translated by W.J.G., pages 53,54).

"It has, it is true, exactly three dimensions; which means that the elements of our visual sensations (those at least which concur in forming the concept of extension) will be completely defined if we know three of them; or, in mathematical language, they will be functions of three independent variables. But let us look at the matter a little closer. The third dimension is revealed to us in two different ways: by the effort of accommodation and by the convergence of the eyes. No doubt these two indications are always in harmony; there is between them a constant relation; or, in mathematical language, the two variables which measure these two muscular sensations do not appear to us as independent......... But that is, so to speak, an experimental fact. Nothing prevents us *a priori* from assuming the contrary, and if the contrary takes place, if these two muscular sensations both vary independently, we must take into account one more independent variable, and complete visual space will appear to us as a physical continuum of four dimensions."

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*This paper was communicated in two parts the first part, immediately after the first communication of Prof. Bryan, and the second part, after his second communication (See p. 1 of this Bulletin).

In my method I may claim that the independent variation of the two muscular sensations would find ample scope, if we could ever so educate ourselves as to acquire this power of independent variation. My method might be a help towards such an education. At any rate, it would place before our eyes a fairly approximate representation of four dimensional figures and be useful to us in the study of four-dimensional geometry.

II

Professor Bryan after all seems to admit that my method is also a possible method of representing stereoscopically a four-dimensional figure, but he says that his method is superior to mine in as much as it depends only on the *single* principle of binocular vision, whereas mine requires the *additional* principle of accomodation. *A priori*, it would seem evident that to produce from the two-dimensional picture on the retina a four-dimensional impression, *two* and not *one* independent physiological adaptations of the eyes are indispensably necessary. I do not, however, see any good in further prolonging the controversy between us. Both of us have fairly stated our methods. It would lie with other mathematicians interested in the problems of four dimensions to accept or reject either.

REVIEW.

Modern Analysis—2nd edition. By E. T. Whittaker and G. N. Watson. Pp. 560. Cambridge University Press, 1915.

This edition of Professor Whittaker's book differs so largely from the first one, not only in the amount of information, but also in the method of treatment, that it can really claim to be a new book.

There are two parts: the first part deals with the processes of Analysis, whilst the second treats of a number of transcendental functions. Eleven chapters have been assigned to each part, and the decimal system of paragraphing has been adopted.

In Part I, the first chapter treats of complex numbers, as in the first edition. But here we have an addition giving us a glimpse of the arithmetic theory of numbers (as propounded by Dedekind), and complex numbers have, accordingly, been introduced as number-pairs (Grössenpaare).

Chapter II treats of the theory of convergence. The theorems on limits have been developed in the light of the modern theory of numbers and additions have been made both as regards these theorems and the tests of convergence. Three articles have been added on double series; giving the condition of its convergence, a statement of Professor Pringsheim's theorem on its summation, and a property of absolutely convergent double series.

The old Chapter IV has been made Chapter III in the present volume. It deals with continuous functions and uniform convergence. A number of very useful theorems on real functions and their continuity have been added to this Chapter.

In Chapter IV of this volume, which is an addition, we find the theory of Riemann integration written by Mr. Watson. This interesting chapter also contains an account of infinite integrals together with an article on double and repeated integrals. But while it awakens the interest of the reader, it does not satisfy him, as it contains only brief sketches.

The matter of the old Chapter III finds a place in Chapter V of the new book. This contains an account of the fundamental properties of Analytic Functions, and has been left untouched except in one or two places. So far as this chapter goes, the treatment is excellent.

Chapter VI is the old Chapter V and deals with the Theory of Residues. Whereas in the previous edition the application of the theorem of residues was illustrated by the solution of definite examples, in the new edition specific rules have been given to deal with types of particular integrals. It is curious that amongst the books of reference at the end of the chapter, there is not a single English book, especially when we have a good treatment of the subject in Cambridge University Tract No. 15 by the joint author, Mr. Watson.

Chapter VII, which treats of the expansion of functions in infinite series, is, save the addition of Weierstrass's factor-theorem and Borél's integral, materially the same as old Chapter VI.

Chapter VIII deals with asymptotic expansions and summable series and contains the matter of old Chapter VIII, together with a few methods of summation of non-convergent series, *viz.*, those of Borél, Euler and Cesàro.

In Chapter IX (corresponding to old Chapter VII) which deals with Fourier's series there are many changes: the whole method of treatment has been recast and a glimpse of the vast subject is held out before us. Thus the authors have given Féjer's theorem, the theorem of Hurwitz-Liapounoff and Riemann's theorem on trigonometrical series. However, it is to be regretted that attention has not been drawn to the historically important fact of the existence of continuous functions whose Fourier series do not converge. This could have been easily done by the mention of the famous example of Prof. Schwarz,* or the example recently given by Prof. Féjer.¶

The last two chapters deal respectively with the theory of linear Differential equations and Integral equations, the author of the latter chapter being Mr. Watson. Differential equations having two or three singularities have only been treated in Chapter X; while an account of the equations of Fredholm and Volterra, in their simplest forms, has been given in Chapter XI. These are quite new chapters and are valuable introductions to two useful branches of higher analysis.

Those who wish to study the theory of functions, can have a good introduction to the higher treatises on the subject, if they go through the first nine chapters of this volume.

Part II begins with Chapter XII, which deals with the Gamma Function. With the exception of two or three theorems, the treatment is materially the same as in the previous edition.

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* See Hobson's, *The theory of Functions of a real-variable and the theory of Fourier's Series*, Art. 465. In this connection, it may be mentioned that the name of such a famous man as Professor Schwarz is wrongly spelt as Schwartz on p. 181.

¶See *Rendiconti d. Circolo Matematico di Palermo*, 1909.

The next chapter, which is an addition by Mr. Watson, treats of the Zeta-function of Riemann. Chapter XIV is on the Hypergeometric Function and does not materially differ from Chapter XI of the previous edition.

Chapters XV and XVII, deal respectively with Legendre and Bessel Functions and are, except in one or two places, practically the same as in the previous edition.

Chapter XVI, dealing with the Confluent Hypergeometric Function, is an addition. The reader finds Kummer's two formulæ, together with an account of the Whittakerian function in this chapter. The parabolic cylinder functions also find a mention here. But, in this connection, the investigation of Prof. Whittaker does not receive more than a passing reference to the original paper.

Chapter XVIII, corresponding to the old Chapter XIII, has been completely remodelled. This treats of the equations of Mathematical Physics, and gives the solutions of Laplace's and allied equations in the forms first given by Prof. Whittaker in *Mathematische Annalen*, Vol. 57, about fourteen years ago. It is a little disappointing to find that neither Prof. Whittaker nor Dr. Bateman has been able to apply Prof. Whittaker's solution to certain famous boundary-value problems, *e.g.*, the problem of wave-propagation in a region bounded by a single ellipsoid of three unequal axes, or a region bounded by two non-concentric spheres.

Chapter XIX contains new matter, and gives the solution of Mathieu's general equation by various methods, including those of Hill, Floquet, Lindemann and Stieltjes. Prof. Whittaker has also given the results of his own investigations on the subject of this Chapter.

Chapter XX deals with the Weierstrassian elliptic functions and contains the matter of Chapters XIV and XVI of the first edition. It may be noted that the treatment is more lucid than in any other English book on the subject; nevertheless amongst the books of reference, one misses the comprehensive treatise on the subject by Dr. Hancock which has also the advantage of being in English.

Chapter XXI, treating of the theta-functions, is an addition. This gives some of the properties of Jacobi's theta-function, the method of inversion and, at the end of the Chapter, in smaller type, a sketch of the vast subject of modular functions which owes its development chiefly to Prof. Felix Klein.

The last Chapter is on the Jacobian elliptic functions and is not materially the same as in the previous edition, while at the end of the book, a useful appendix dealing with the elementary transcendental functions, has been added.

Although the present edition is a little disappointing in certain respects, it is bound to prove very useful to those Indian students who intend to go in for research work. The style is fascinating and lucid, the authors deduce their theorems almost from the first principles and, above all, they have given a judicious arrangement of the subject matter.

HARIPROSANNA BANERJEA.